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MARTINGALE-LIKE PROCESSES

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled MARTINGALE-LIKE PROCESSES submitted by BYRON ALLAN SCHMULAND in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

A real-valued random process $\{X_n\}$ is called a martingale if $E(X_{n+1}|\mathcal{B}_n) = X_n$ for all $n \in \mathbb{N}$. In this thesis, we examine some martingale-like processes and study their properties.

In the third chapter we explore the relationships between the processes under consideration. Also, a number of established convergence results are presented. We then turn to the eventual martingale, and conditions are given under which it converges almost surely. Next, the behavior of the given processes under the weak Riesz decomposition and under stopping times is considered.

In the fourth and fifth chapters, the martingale-like processes are generalized, respectively, to processes indexed by $\mathbb{N}_+^2$ and to Banach-valued processes.

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Finally, I am most grateful to my parents for their love and support throughout nearly twenty years of schooling.

E quindi uscimmo a riveder le stelle

- Dante

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CHAPTER 1

INTRODUCTION

A sequence of adapted, integrable random variables $\{X_n\}$ defined on a filtered probability space $(\Omega, \mathcal{B}, P, \{\mathcal{B}_n\})$ is called a martingale if $E(X_{n+1}|\mathcal{B}_n) = X_n$ a.s. for each $n \in \mathbb{N}$. The martingale has been an object of study for a number of years, and has been shown to have some remarkable properties. For any given martingale property the question arises: How closely must a process $\{X_n\}$ resemble a martingale in order to preserve that property? Over the years, in an attempt to answer questions of this type, various martingale-like processes have been developed and studied. In this thesis we investigate the properties of the following types of martingale-like processes; quasimartingales, amarts, progressive martingales, martingales in the limit, L^1-martingales and games which get fairer with time.

The second chapter contains most of the important definitions and notation used throughout the thesis. A number of results on conditional expectation and uniform integrability are also given.

We begin the third chapter by defining the above processes, which are then compared to determine which imply which. In Section 3.3, a number of known convergence results are presented for martingales and martingale-like processes. The remainder of the section is devoted to the convergence properties of eventual martingales. In Section 3.4 we look at the weak Riesz decomposition and show which of the processes under consideration have this property. This decomposition is then

used to prove an amart convergence theorem. In Section 3.5 the behavior of the martingale-like processes under stopping times and transforms is considered.

In Chapters 4 and 5 the above martingale-like processes are generalized, respectively, to process indexed by $\mathbb{N}_+^2$ and to processes taking values in a separable Banach space. The convergence of such generalized processes is then considered.

Throughout this thesis, for the sake of completeness, we have included as many proofs as possible.

CHAPTER 2

PRELIMINARIES

2.1 Definitions and Notation

Let Ω be a set of points. A non-empty class $\mathcal{B}$ of subsets of Ω is called a *σ-algebra* if

(i) $B \in \mathcal{B} \Rightarrow B^c \in \mathcal{B}$

(ii) if $\{B_n\}$ is a countable sequence in $\mathcal{B}$ then $\cup B_n \in \mathcal{B}$

If $\{B_\alpha, \alpha \in \Gamma\}$ is a non-empty class of subsets of Ω, then $\sigma(B_\alpha, \alpha \in \Gamma)$ denotes the smallest σ-algebra containing each B_α.

If $A, B \in \mathcal{B}$, their symmetric difference $(A^c \cap B) \cup (A \cap B^c)$, is denoted $A \Delta B$.

A function $P: \mathcal{B} \to [0,1]$ is called a *probability measure* if

(i) $P(\Omega) = 1$

(ii) if $\{B_n\}$ is a countable sequence of disjoint sets in $\mathcal{B}$, then $P(\cup B_n) = \Sigma P(B_n)$.

If $\{\mathcal{B}_n\}$ is an increasing sequence of sub σ-algebras of $\mathcal{B}$, then we call $(\Omega, \mathcal{B}, P, \{\mathcal{B}_n\})$ a *filtered probability space* and $\{\mathcal{B}_n\}$ is called a *filtration*. The elements of $\mathcal{B}$ are referred to as *events*.

If $X: \Omega \to \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ is a function, we let $\sigma(X) = \sigma(\{\omega | X(\omega) \le x\} | x \in \mathbb{R})$. If X is $\mathcal{B}$-measurable, that is if $\sigma(X) \subseteq \mathcal{B}$, then X is called a *random variable*. Two random variables X and Y are said to be equivalent if $P[X=Y] = 1$. We rarely distinguish between an equivalence class formed by this relation and a particular random

variable in that class. If $B \in \mathcal{B}$ we define 1_B, the indicator function of B, as follows

$$1_B(\omega) = \begin{cases} 1 & \omega \in B \\ 0 & \omega \notin B \end{cases}$$

Unless otherwise stated, the index set for our random processes is $\mathbb{N} = \{1,2,3,\ldots\}$. When it is more convenient to add a zero element we shall use $\mathbb{N}_+ = \{0,1,2,\ldots\}$.

A *stopping time* is a random variable $\tau: \Omega \to \mathbb{N} \cup \{\infty\}$ such that $\{\omega | \tau(\omega) = n\} \in \mathcal{B}_n$ for all $n \in \mathbb{N}$. If $\{X_n\}$ is a sequence of random variables, the random variable X_τ is defined as follows:

$$X_\tau(\omega) = \begin{cases} (X_{\tau(\omega)})(\omega) & \tau(\omega) < \infty \\ 0 & \tau(\omega) = \infty \end{cases}$$

The σ-algebra associated with τ, denoted $\mathcal{B}_\tau$ is defined as $\mathcal{B}_\tau = \{B \in \sigma(\cup \mathcal{B}_n) | B \cap \{\tau = n\} \in \mathcal{B}_n$ for each $n \in \mathbb{N}\}$. The set of all bounded stopping times is denoted by T and is given the natural ordering $(<)$, if $\tau_1, \tau_2 \in T$ and $P[\tau_2 < \tau_1] = 0$ then $\tau_1 < \tau_2$.

A random variable is called *simple* if there are finitely many numbers $x_1, \ldots, x_n \in \mathbb{R}$ such that $\Sigma P[X = x_i] = 1$.

The integral of a simple random variable is defined as

$$\int X dP = \sum x_i P[X = x_i]$$

If X is a positive ($\equiv$ non-negative) random variable, its integral is defined as

$$\int XdP = \lim_{n} \uparrow \int X_n dP$$

where $\{X_n\}$ is a sequence of simple random variables which increases pointwise to X. If the above limit is finite, X is said to be integrable. An arbitrary random variable X is said to be *integrable* if both its positive and negative parts, $X^+ = X1_{\{X \geq 0\}}$ and $X^- = -X1_{\{X \leq 0\}}$ are integrable, and we define

$$\int X = \int XdP = \int X^+ dP - \int X^- dP$$

We sometimes write $E(X)$, the expected value of X, for $\int XdP$. If $B \in \mathcal{B}$ we define $\int_B X = \int X1_B$.

The set of all integrable random variables forms a Banach space, with the norm $||X||_1 = E(|X|)$. This space is denoted by $L^1(\Omega,\mathcal{B},P)$, written L^1 for short. A family $\{X_\alpha, \alpha \in \Gamma\}$ in L^1 is called bounded if $\sup_{\alpha\in\Gamma} ||X_\alpha||_1 < \infty$.

A random variable is called *bounded* if there exists $a \in \mathbb{R}$ such that $P[|X| \geq a] = 0$. The set of all bounded random variables forms a Banach space, with the norm $||X||_\infty = \inf\{a \in \mathbb{R} \,|P[|X| \geq a] = 0\}$. This space is denoted by $L^\infty(\Omega,\mathcal{B},P)$, written L^∞ for short.

A sequence of random variables $\{X_n\}$ on $(\Omega,\mathcal{B},P,\{\mathcal{B}_n\})$ is called *adapted* if X_n is $\mathcal{B}_n$-measurable for each $n \in \mathbb{N}$.

A sequence of random variables $\{X_n\}$ is said to converge *almost surely* (a.s.) if $P[\lim X_n(\omega)$ exists and is finite$] = 1$. $\{X_n\}$ converges to X *in probability* if for each $\varepsilon > 0$, the sequence $P[|X_n-X| \geq \varepsilon]$ converges to zero as $n \to \infty$. $\{X_n\}$ converges to X *in L^1* if $||X_n-X||_1$ converges to zero as $n \to \infty$. $\{X_n\}$ is said to

converge to X *weakly in* L^1 if $E(X_n \cdot Y) \to E(X \cdot Y)$ for each $Y \in L^\infty$.

Unless otherwise stated, in what follows we assume that we have a fixed filtered probability space $(\Omega, \mathcal{B}, P, \{\mathcal{B}_n\})$ with $\mathcal{B} = \sigma(\cup \mathcal{B}_n)$. Unless otherwise stated, a sequence of random variables $\{X_n\}$ is assumed to be adapted and in L^1.

2.2 Conditional Expectation

DEFINITION 2.2.1. Let $X \in L^1$ and $\mathcal{C}$ be a sub σ-algebra of $\mathcal{B}$. By the Radon-Nikodym theorem there exists a $\mathcal{C}$-measurable random variable which we denote by $E(X|\mathcal{C})$, uniquely determined except on an event of probability zero, such that

$$\int_C X = \int_C E(X|\mathcal{C})$$

for each $C \in \mathcal{C}$. $E(X|\mathcal{C})$ is called the *conditional expectation* of X given $\mathcal{C}$.

□

The following propositions give some of the properties of the conditional expectation operator. The proofs can be found in almost any probability text, for example [32].

PROPOSITION 2.2.2. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$ are two sub σ-algebras of $\mathcal{B}$ then $E(E(X|\mathcal{C}_1)|\mathcal{C}_2) = E(E(X|\mathcal{C}_2)|\mathcal{C}_1) = E(X|\mathcal{C}_1)$.

PROPOSITION 2.2.3. If $X, Y, XY \in L^1$ and Y is $\mathcal{C}$-measurable then $E(XY|\mathcal{C}) = YE(X|\mathcal{C})$.

PROPOSITION 2.2.4. If X and Y are independent then $E(X|\sigma(Y)) = E(X)$.

PROPOSITION 2.2.5. Let $\{X_n\}$ be a non-decreasing sequence of non-

negative random variables such that $X_n \to X$ a.s. and $X \in L^1$. Then $E(X_n|\mathcal{C}) \to E(X|\mathcal{C})$ a.s.

PROPOSITION 2.2.6. Let g be a convex function on $\mathbb{R}$ and let $X \in L^1$ such that $g(X) \in L^1$. Then

$$g(E(X|\mathcal{C})) \leq E(g(X)|\mathcal{C})$$

2.3 Uniform Integrability

The notion of the uniform integrability of a family of random variables is very important when considering the convergence properties of martingales and martinale-like sequences. For that reason, we present the main results on uniform integrability in this section.

DEFINITION 2.3.1. A family $\{X_\alpha, \alpha \in \Gamma\}$ of random variables in L^1 is called *uniformly integrable* if

$$\lim_{a\uparrow\infty} \sup_{\alpha\in\Gamma} \int_{\{|X_\alpha|\geq a\}} |X_\alpha| = 0$$

PROPOSITION 2.3.2. $\{X_\alpha\}$ is uniformly integrable if and only if $\{X_\alpha\}$ is bounded in L^1 and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $B \in \mathcal{B}$ with $P(B) \leq \delta$ then

$$\sup_{\alpha\in\Gamma} \int_B |X_\alpha| \leq \varepsilon$$

PROOF ([20]): ($\Rightarrow$) Suppose $\{X_\alpha\}$ is uniformly integrable and $\varepsilon > 0$. Choose $a \in \mathbb{R}$ so that

$$\sup_{\alpha\in\Gamma} \int_{\{|X_\alpha|\geq a\}} |X_\alpha| \leq \varepsilon/2$$

Then for each $B \in \mathcal{B}$ we have that

$$\sup_{\alpha\in\Gamma} \int_B |X_\alpha| \leq a\cdot P(B) + \sup_{\alpha\in\Gamma} \int_{\{|X_\alpha|\geq a\}} |X_\alpha|$$

If we let $B = \Omega$ we see that $\sup_{\alpha\in\Gamma} E(|X_\alpha|) \leq a + \varepsilon/2 < \infty$ so that $\{X_\alpha\}$ is bounded in L^1.

Further, if $P(B) < \delta = \varepsilon/2a$ we get $\sup_{\alpha\in\Gamma} \int_B |X_\alpha| \leq \varepsilon$.

(<=) Now suppose that the two conditions hold. Given $\varepsilon > 0$ let $\delta > 0$ as in the condition, and let $a = \sup E(|X_\alpha|)/\delta$. For each $\alpha \in \Gamma$, $P[|X_\alpha| \geq a] \leq \delta$ so that

$$\sup_{\alpha\in\Gamma} \int_{\{|X_\alpha|\geq a\}} |X_\alpha| \leq \varepsilon$$

□

PROPOSITION 2.3.3. If $\{X_\alpha\}$ is uniformly integrable. Then the L^1 closure of $\{X_\alpha\}$, denoted by $c\ell(\{X_\alpha\})$, is uniformly integrable.

PROOF ([20]): Suppose $\{X_\alpha\}$ is uniformly integrable. By 2.3.2 $\{X_\alpha\}$ is L^1-bounded, so clearly $c\ell(\{X_\alpha\})$ is L^1-bounded.

Let $\varepsilon > 0$ and let $\delta > 0$, such that $P(B) \leq \delta$ implies

$$\sup_{\alpha\in\Gamma} \int_B |X_\alpha| \leq \varepsilon$$

Now let $X \in c\ell(\{X_\alpha\})$. Then for each $\alpha \in \Gamma$ we have

$$\int_B |X| \leq \int_B |X_\alpha| + ||X-X_\alpha||_1 \leq \varepsilon + ||X-X_\alpha||_1$$

Since $X \in c\ell(\{X_\alpha\})$, we can choose $\alpha \in \Gamma$ so that $||X-X_\alpha||_1$ is as close to zero as we want, and thus we see that $\int_B |X| \leq \varepsilon$. By applying 2.3.2 to $c\ell(\{X_\alpha\})$ we see that $c\ell(\{X_\alpha\})$ is uniformly integrable. □

PROPOSITION 2.3.4. $\{X_\alpha\}$ is uniformly integrable if and only if there exists a function G defined on $\mathbb{R}_+$, positive, increasing, convex and such that $\lim_{x\to\infty} G(x)/x = \infty$ and

$$\sup_{\alpha\in\Gamma} E(G\circ|X_\alpha|) < \infty$$

PROOF ([20]): ($\Leftarrow$) Suppose that the function G described above exists. Let $\varepsilon > 0$ and let $d = M/\varepsilon$ where $M = \sup_{\alpha\in\Gamma} E(G\circ|X_\alpha|)$. Choose a so large that $G(x)/x > d$ if $x > a$. Thus on the set $\{|X_\alpha| \geq a\}$ we have $|X_\alpha| \leq G\circ|X_\alpha|/d$. Therefore

$$\sup_{\alpha\in\Gamma} \int_{\{|X_\alpha|\geq a\}} |X_\alpha| \leq \frac{1}{d} \sup_{\alpha\in\Gamma} \int_{\{|X_\alpha|\geq a\}} G\circ|X_\alpha| \leq \frac{M}{d} = \varepsilon$$

so that by definition, $\{X_\alpha\}$ is uniformly integrable.

($\Rightarrow$) Define $a_n(X) = P[|X| > n]$ for $n \in \mathbb{N}$.

Let $g_o = 0$ and let $\{g_n\}$ be a sequence of constants which increases to infinity. Let $g = \sum_{n=0}^{\infty} (g_n \cdot 1_{[n,n+1]})$ and let $G(x) = \int_0^x g(t)dt$. Then we have

$$\begin{aligned} E(G\circ|X|) &\leq g_1 P[1<|X|\leq 2] + (g_1+g_2)P[2<|X|\leq 3] + \ldots \\ &= \sum_{n=1}^{\infty} g_n \cdot a_n(X) \end{aligned}$$

Now G is positive, increasing, convex and $\lim G(x)/x = \infty$. If we can find a sequence $\{g_n\}$ so that the sum $\Sigma g_n \cdot a_n(X)$ is uniformly bounded for $X \in \{X_\alpha, \alpha\in\Gamma\}$ then we are done. Pick an increasing sequence $\{c_n\}$, $c_n \uparrow \infty$ and

$$\sup_{\alpha\in\Gamma} \int_{\{|X_\alpha|\geq c_n\}} |X_\alpha| \leq 1/2^n$$

which can be done, by the hypothesis of uniform integrability. Then

$$\int_{\{|X_\alpha|\geq c_n\}} |X_\alpha| \geq \sum_{k=c_n}^{\infty} k \cdot P[k<|X_\alpha|\leq k+1] \geq \sum_{m=c_n}^{\infty} P[|X_\alpha|>m]$$

$$= \sum_{m=c_n}^{\infty} a_m(X_\alpha)$$

For $m \in \mathbb{N}$ set g_m = "the number of c_n's less than m." We then get

$$\sum_{n=1}^{\infty} g_n \cdot a_n(X_\alpha) = \sum_n \sum_{m=c_n}^{\infty} a_m(X_\alpha) \leq 1 \text{ for each } \alpha \in \Gamma.$$

□

PROPOSITION 2.3.5. If $X \in L^1$ then the following family of random variables is uniformly integrable,

$$\{E(X|\mathcal{C})\,|\,\mathcal{C} \text{ is a sub } \sigma\text{-algebra of } \mathcal{B}\}.$$

PROOF: The trivial family $\{X\}$ is uniformly integrable as $X \in L^1$. Let G be as in 2.3.4 for this family. Noting that G is non-decreasing and convex, and also that the absolute value function is convex, and by employing 2.2.6 we get

$$\sup_{\mathcal{C}} E(G \circ |E(X|\mathcal{C})|) \leq \sup_{\mathcal{C}} E(G \circ E(|X|\,|\mathcal{C}))$$

$$\leq \sup_{\mathcal{C}} E(E(G \circ |X|\,|\mathcal{C})) = \sup_{\mathcal{C}} E(G \circ |X|) < \infty$$

By 2.3.4 the given family is uniformly integrable.

□

PROPOSITION 2.3.6. If $\{X_n\}$ is a uniformly integrable sequence which converges to X a.s., then $X_n \to X$ in L^1.

PROOF ([25]): For $a > 0$ define $f_a(x) = \begin{cases} x & |x| < a \\ a\cdot\mathrm{sgn}(x) & \text{otherwise} \end{cases}$

This function satisfies $|x-f_a(x)| \leq |x|$ for each $x \in \mathbb{R}$. By the triangle inequality we get, for any $m, n \in \mathbb{N}$

$$E|X_n-X_m| \leq E|f_a(X_n)-f_a(X_m)| + E|X_m-f_a(X_m)| + E|X_n-f_a(X_n)|$$

If $X_n \to X$ a.s., then $f_a(X_n) \to f_a(X)$ a.s. . Since f_a is a bounded, continuous function, and as the random variables $f_a(X_n)$ are dominated by the constant a, the dominated convergence theorem shows that $f_a(X_n)$ converges to $f_a(X)$ in L^1. On the other hand,

$$E|X_n-f_a(X_n)| \leq \int_{\{|X_n|\geq a\}} |X_n|$$

by the definition of f_a. By letting $n, m \to \infty$ and then $a \to \infty$ in the first inequality we see that $\{X_n\}$ is Cauchy, and therefore converges to X in L^1.

CHAPTER 3

REAL-VALUED PROCESSES INDEXED BY $\mathbb{N}$

3.1 Background

The term martingale, in the gambling context, dates back to the early nineteenth century, but work in what is now known as martingale theory began less than sixty years ago. Of major importance in this area is the work of Doob, done in the 1940's and early 1950's. His 1953 book "Stochastic Processes" [11] has remained a standard reference for thirty years. Today, martingales form an active area of research which occupies a prominent position in the field of probability theory.

In an attempt to generalize the concept of martingales, a number of martingale-like sequences have been defined. One of the first, called a quasimartingale, was introduced in 1965 by Fisk [16]. In 1970 Blake [4] gave us "games which get fairer with time," while in the same year Alloin [1] introduced "progressive martingales." In 1973 Mucci [22] came up with "martingales in the limit." 1974 brought "asymptotic martingales" or "amarts" in a paper by Austin, Edgar and Tulcea [2], based on an earlier idea by Meyer [21]. In 1975 Tomkins [31] introduced "eventual martingales."

In Section 3.2 we shall define and compare the above martingale-like processes.

Given the importance of Doob's martingale convergence theorem, it is not surprising that most attempts at generalizing martingales are accompanied by some convergence results. In Sections 3.3 and 3.4 we explore the convergence properties of the above processes.

3.2 Relationships between the processes

We remind the reader that we have a fixed filtered probability space $(\Omega,\mathcal{B},P,\{\mathcal{B}_n\})$ and that all of the following processes are assumed to be on this space and adapted to it, as well as integrable. To emphasize the particular filtration used, we sometimes write the process as $\{X_n,\mathcal{B}_n\}$ rather than just $\{X_n\}$. All equalities, inequalities and set containments used in the following definitions can be taken to be in the a.s. sense.

DEFINITION 3.2.1 ([25]). A process $\{X_n\}$ is called a *sub* [resp. *super*] *martingale* if $E(X_{n+1}|\mathcal{B}_n) \geq$ [resp. $\leq$]X_n for each $n \in \mathbb{N}$. In the case of equality, $\{X_n\}$ is called a *martingale*.

DEFINITION 3.2.2 ([31]). A process $\{X_n\}$ is called an *eventual supermartingale* if $P[\liminf\{E(X_{n+1}|\mathcal{B}_n) \leq X_n\}] = 1$. In the case of equality, $\{X_n\}$ is called an *eventual martingale*.

DEFINITION 3.2.3 ([4]). A process $\{X_n\}$ is called a *game which gets fairer with time* [abbrev. GFT] if for each $\varepsilon > 0$ the sequence $\{y_m\}$ converges to zero, where

$$y_m = \sup_{n\geq m} P[|E(X_n-X_m|\mathcal{B}_m)| > \varepsilon]$$

DEFINITION 3.2.4 ([26]). A process $\{X_n\}$ is called an *L^1-martingale* if the sequence $\{y_m\}$ converges to zero, where

$$y_m = \sup_{n\geq m} \|E(X_n-X_m|\mathcal{B}_m)\|_1$$

DEFINITION 3.2.5 ([13]). A process $\{X_n\}$ is called an *amart* if the net

$\{E(X_\tau), \tau \in T\}$ converges in $\mathbb{R}$.

DEFINITION 3.2.6 ([22]). A process $\{X_n\}$ is called a *martingale in the limit* [abbrev. MIL] if $\{Y_m\}$ converges to zero almost surely where

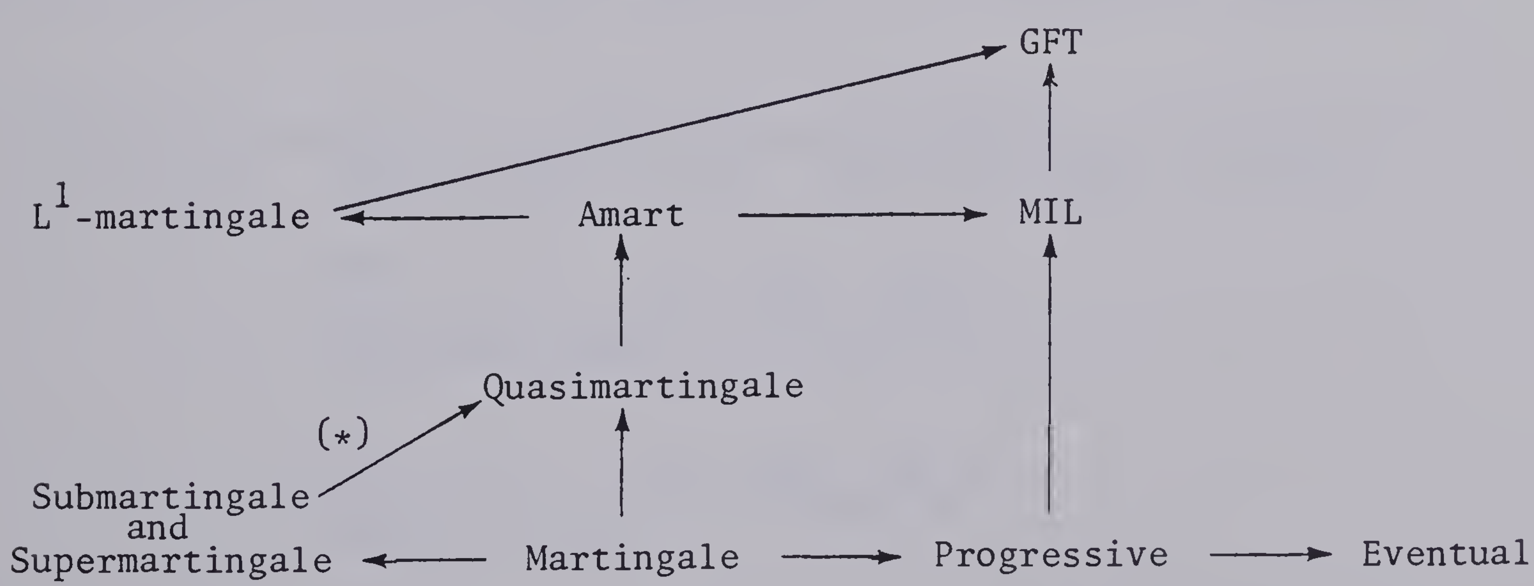
$$Y_m = \sup_{n \geq m} |E(X_n - X_m | \mathcal{B}_m)|$$

DEFINITION 3.2.7 ([1]). A process $\{X_n\}$ is called a *progressive martingale* if $\{E(X_n|\mathcal{B}_{n-1}) = X_{n-1}\} \subseteq \{E(X_{n+1}|\mathcal{B}_n) = X_n\}$ for $n > 1$ and $\lim_n P[E(X_{n+1}|\mathcal{B}_n) = X_n] = 1$.

DEFINITION 3.2.8 ([16]). A process $\{X_n\}$ is called a *quasimartingale* if $\sum_n \| E(X_{n+1} - X_n | \mathcal{B}_n) \|_1 < \infty$.

The following diagram illustrates some of the relationships between the above martingale-like processes. A number of the non-trivial proofs follow.

DIAGRAM 3.2.9.



(*) if and only if $\{X_n\}$ is L^1-bounded

PROPOSITION 3.2.10. A super or submartingale is a quasimartingale if and only if it is L^1-bounded.

PROOF: Let $\{X_n\}$ be a submartingale. Then

$$\sum_n E(|E(X_{n+1}-X_n|\mathcal{B}_n)|) = \sum_n E(E(X_{n+1}-X_n|\mathcal{B}_n))$$

$$= \sum_n (E(X_{n+1})-E(X_n))$$

$$= \lim_n E(X_n) - E(X_1)$$

The increasing sequence $\{E(X_n)\}$ has a finite limit if and only if $\{X_n\}$ is L^1-bounded. The corresponding result for supermartingales is obtained by considering $\{-X_n\}$.

□

PROPOSITION 3.2.11. Every quasimartingale is an amart.

PROOF ([13]): Let $\{X_n\}$ be a quasimartingale and let $\varepsilon > 0$. Choose N so that $\sum_N^\infty E(|E(X_{n+1}-X_n|\mathcal{B}_n)|) \leq \varepsilon$. Let $\tau \in T$, $\tau > N$. Since τ is bounded we can find $M \in \mathbb{N}$ such that $\tau < M$, and then we consider

$$|\int (X_\tau - X_M)| = |\sum_{k=N}^{M} \int_{\{\tau=k\}} (X_k - X_M)| = |\sum_{k=N}^{M} \sum_{n=k}^{M-1} \int_{\{\tau=k\}} (X_n - X_{n+1})|$$

$$= |\sum_{n=N}^{M-1} \sum_{k=N}^{n} \int_{\{\tau=k\}} X_n - E(X_{n+1}|\mathcal{B}_n)|$$

$$\leq \sum_{n=N}^{M-1} \sum_{k=N}^{n} \int_{\{\tau=k\}} |X_n - E(X_{n+1}|\mathcal{B}_n)|$$

$$\leq \sum_{n=N}^{\infty} E|E(X_{n+1}-X_n|\mathcal{B}_n)| \leq \varepsilon.$$

If $\tau_1, \tau_2 \in T$ and $\tau_1 > N$ and $\tau_2 > N$, choose $M \in \mathbb{N}$ so that $M > \tau_1$ and $M > \tau_2$. Then

$$|\int X_{\tau_1} - \int X_{\tau_2}| \leq |\int X_{\tau_1} - X_M| + |\int X_M - \int X_{\tau_2}| \leq 2\varepsilon$$

so that the net $\{E(X_\tau), \tau \in T\}$ converges and hence $\{X_n\}$ is an amart. $\square$

PROPOSITION 3.2.12. Every amart is a MIL.

PROOF ([14]): Let $\{X_n\}$ be an amart and suppose that it is not a MIL. Then there exists $\delta > 0$ such that

$$P[\limsup_m (\sup_{n \geq m} |E(X_n - X_m | \mathcal{B}_m)|) > \delta] > 2\delta$$

Therefore either

$$\text{(i)} \quad P[\limsup_m (\sup_{n \geq m} (E(X_n - X_m | \mathcal{B}_m))) > \delta] > \delta$$

or

$$\text{(ii)} \quad P[\limsup_m (\sup_{n \geq m} (E(X_m - X_n | \mathcal{B}_m))) > \delta] > \delta$$

Since $\{X_n\}$ may be replaced by $\{-X_n\}$, it is without loss of generality that we assume (i) holds. Let $N \in \mathbb{N}$. Then

$$P[\text{there exist } m,n \,|\, N \leq m \leq n \text{ and } E(X_n - X_m | \mathcal{B}_m) > \delta] > \delta$$

and thus there exists $N' \in \mathbb{N}$ such that

$$P[\text{there exist } m,n \,|\, N \leq m \leq n \leq N' \text{ and } E(X_n - X_m | \mathcal{B}_m) > \delta] > \delta$$

Now for each m such that $N \leq m \leq N'$ define the set A_m as follows

$$A_m = \{\text{there exists } n \mid m \le n \le N' \text{ and } E(X_n - X_m \mid \mathcal{B}_m) > \delta\}$$

and define $A = \bigcup_{m=N}^{N'} A_m$.

Thus $A_m \in \mathcal{B}_m$ for each m and $P(A) > \delta$. Define two stopping times τ_1 and τ_2 by

$$\tau_1(\omega) = \begin{cases} \min\{m\colon N \le m \le N',\ \omega \in A_m\} & \omega \in A \\ N' & \omega \notin A \end{cases}$$

$$\tau_2(\omega) = \begin{cases} \min\{n\colon m \le n \le N',\ E(X_n - X_m \mid \mathcal{B}_m) > \delta\} & \omega \in \{\tau_1 = m\} \cap A \\ N' & \omega \notin A \end{cases}$$

Since for any given $m \le n$, the set $\{\tau_1 = m\} \cap \{\tau_2 = n\} \in \mathcal{B}_m$ we have that

$$\begin{aligned} E(X_{\tau_2}) - E(X_{\tau_1}) &= \sum_{m=N}^{N'} \sum_{n=m}^{N'} E\left((X_n - X_m) 1_{\{\tau_1 = m\} \cap \{\tau_2 = n\}}\right) \\ &= \sum_{m=N}^{N'} \sum_{n=m}^{N'} E\left(E(X_n - X_m \mid \mathcal{B}_m) 1_{\{\tau_1 = m\} \cap \{\tau_2 = n\}}\right) \\ &\ge \sum_{m=N}^{N'} \sum_{n=m}^{N'} E\left(\delta \cdot 1_{\{\tau_1 = m\} \cap \{\tau_2 = n\}}\right) \ge \delta \cdot P(A) \ge \delta^2 \end{aligned}$$

Hence the net $\{E(X_\tau), \tau \in T\}$ does not converge, so $\{X_n\}$ is not an amart. This is a contradiction so that $\{X_n\}$ must be a MIL. □

PROPOSITION 3.2.13. Every amart is an L^1-martingale.

PROOF: Let $\{X_n\}$ be an amart and let $m \in \mathbb{N}$ so that if $\tau_1, \tau_2 \in T$ and $\tau_1 > m$ and $\tau_2 > m$ then

$$|E(X_{\tau_2}) - E(X_{\tau_1})| \leq \varepsilon$$

Let $n \geq m$ and let $A = \{E(X_n - X_m | \mathcal{B}_m) > 0\}$. Define two stopping times by

$$\tau_1(\omega) = \begin{cases} m & \omega \notin A \\ n & \omega \in A \end{cases} \qquad \tau_2(\omega) = \begin{cases} m & \omega \in A \\ n & \omega \notin A \end{cases}$$

Then

$$\begin{aligned} \|E(X_n - X_m | \mathcal{B}_m)\|_1 &= \int |E(X_n - X_m | \mathcal{B}_m)| \\ &= \int_A E(X_n - X_m | \mathcal{B}_m) - \int_{A^c} E(X_n - X_m | \mathcal{B}_m) \\ &= \int_A (X_n - X_m) - \int_{A^c} (X_n - X_m) \\ &= \int (X_{\tau_1} - X_m) - \int (X_{\tau_2} - X_m) \\ &\leq 2\varepsilon. \end{aligned}$$

Thus $\{X_n\}$ is an L^1-martingale.

□

PROPOSITION 3.2.14. Every progressive martingale is a MIL.

PROOF: Let $\{X_n\}$ be a progressive martingale and let $A_n = \{E(X_{n+1} | \mathcal{B}_n) = X_n\}$ for each $n \in \mathbb{N}$. By hypothesis $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$ and $\lim P(A_n) = 1 = P(\lim A_n)$. Pick $N \in \mathbb{N}$. Then if $N \leq m \leq n$ we get

$$E(X_n - X_m | \mathcal{B}_m) 1_{A_N} = \sum_{k=m}^{n-1} E\left(E(X_{k+1} - X_k | \mathcal{B}_k) 1_{A_k \cap A_N} \middle| \mathcal{B}_m \right) = 0$$

since $A_N \subseteq A_k$ for $k \geq N$ and $A_N \in \mathcal{B}_N \subseteq \mathcal{B}_k$ for $k \geq N$. The fact

that $\{X_n\}$ is a MIL follows, since $P(\lim A_n) = 1$. □

The following examples illustrate, in a negative sense, the relationships between the martingale-like processes. These examples, combined with Diagram 3.2.9 provide a complete picture of these relationships.

EXAMPLE 3.2.15. The deterministic sequence $X_n = 1/n$ is a supermartingale since it is decreasing, and it is clearly L^1-bounded. However since $\{X_n\}$ is not eventually constant, it is not an eventual martingale. $Y_n = -X_n$ provides an L^1-bounded submartingale which is not an eventual martingale.

EXAMPLE 3.2.16. The deterministic sequence $X_n = (-1)^n \frac{1}{2^n}$ is a quasimartingale since

$$\sum_n E|E(X_{n+1}-X_n|\mathcal{B}_n)| = \sum_n E|X_{n+1}-X_n| \leq \sum_n \frac{1}{2^{n+1}} + \frac{1}{2^n} < \infty$$

Clearly $\{X_n\}$ is neither a submartingale nor a supermartingale.

EXAMPLE 3.2.17. The deterministic sequence $X_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k}$ is an amart, with the trivial filtration $\mathcal{B}_n = \mathcal{B} = \{\phi,\Omega\}$, since $\sum(-1)^k \frac{1}{k}$ converges. But since

$$\sum_n E|X_{n+1}-X_n| = \sum_n \frac{1}{n+1} = \infty$$

$\{X_n\}$ is not a quasimartingale.

EXAMPLE 3.2.18. Let $\Omega = [0,1)$ endowed with Lebesque measure,

$\mathcal{B}$ = {all Borel sets}. (Hereafter, this space shall be referred to as the Lebesque space.)

Give it the filtration $\mathcal{B}_j = \mathcal{B}$ for each $j \in \mathbb{N}$. Let

$$X_j = 1_{\left[\frac{j-2^n}{2^n},\ \frac{j+1-2^n}{2^n}\right)} \qquad j = 2^n,\ldots,2^{n+1}-1 \quad n \in \mathbb{N}$$

Thus $\{X_j\}$ is an L^1-martingale as it converges in L^1 to zero, but it is not a MIL as it does not converge pointwise.

EXAMPLE 3.2.19. Let $\Omega = \mathbb{N}$, $\mathcal{B}_n = \mathcal{B}$ = all subsets of $\mathbb{N}$, $P(n) = 1/2^n$ and

$$X_n = \left(\sum_{k=1}^{n} 1_{\{k\}} 2^k\right) + \left(1 - \frac{1}{n}\right) 1_{[n+1,\infty)}$$

for each $n \in \mathbb{N}$. Since $\{E(X_n - X_{n-1}|\mathcal{B}_{n-1}) = 0\} = \{X_n = X_{n-1}\} = \{1,2,3,\ldots,n-1\}$ we have $\{E(X_n - X_{n-1}|\mathcal{B}_{n-1}) = 0\} \subseteq \{E(X_{n+1} - X_n|\mathcal{B}_n)\}$ for $n > 1$, and also $\lim_n P[E(X_n - X_{n-1}|\mathcal{B}_{n-1}) = 0] = 1$. Thus $\{X_n\}$ is a progressive martingale. But

$$\begin{aligned}
\|E(X_{n+1} - X_n|\mathcal{B}_n)\|_1 &= E(X_{n+1}) - E(X_n) \\
&= \left(2^{n+1} - (1 - \tfrac{1}{n})\right) \frac{1}{2^{n+1}} + \left(\frac{1}{n} - \frac{1}{n+1}\right) \sum_{k=n+2}^{\infty} \frac{1}{2^k} \\
&= 1 - \frac{1}{2^{n+1}} \left(\frac{2}{n} - \frac{1}{n+1} - 1\right) \\
&> \frac{1}{2} \quad \text{for } n \in \mathbb{N}
\end{aligned}$$

So that $\{X_n\}$ cannot be an L^1-martingale.

EXAMPLE 3.2.20. For an example of an eventual martingale which is not GFT, see 3.3.13. $\{X_n\}$ cannot be GFT as it is uniformly integrable and does not converge in L^1.

3.3 Convergence

In this section we will consider the convergence properties of the various martingale-like sequences. We will begin by presenting the proofs of three well-known convergence results for martingales. They are as follows: that a positive supermartingale converges a.s., that an L^1-bounded martingale converges a.s., and that a uniformly integrable martingale converges a.s. and in L^1.

LEMMA 3.3.1. If $\{X_n, \mathcal{B}_n\}$ and $\{Y_n, \mathcal{B}_n\}$ are two positive supermartingales and if τ is a stopping time such that $X_\tau \geq Y_\tau$ on $\{\tau < \infty\}$, then if we define

$$Z_n = \begin{cases} X_n & \text{on } \{n < \tau\} \\ Y_n & \text{on } \{n \geq \tau\} \end{cases}$$

$\{Z_n, \mathcal{B}_n\}$ is a positive supermartingale.

PROOF ([25]): Since we can write $Z_n = X_n 1_{\{n<\tau\}} + Y_n 1_{\{n\geq\tau\}}$, we see that Z_n is $\mathcal{B}_n$-measurable for each $n \in \mathbb{N}$.

Since $\{X_n\}$ and $\{Y_n\}$ are supermartingales we get

$$\begin{aligned} Z_n &= X_n 1_{\{\tau>n\}} + Y_n 1_{\{\tau\leq n\}} \\ &\geq E(X_{n+1}|\mathcal{B}_n) 1_{\{\tau>n\}} + E(Y_{n+1}|\mathcal{B}_n) 1_{\{n\geq\tau\}} \\ &= E(X_{n+1} 1_{\{\tau>n\}} + Y_{n+1} 1_{\{n\geq\tau\}} | \mathcal{B}_n) \end{aligned}$$

By the assumption that $X_\tau \geq Y_\tau$ on $\{\tau < \infty\}$ we see that $X_{n+1} \geq Y_{n+1}$ on $\{\tau = n+1\}$ so it follows that

$$1_{\{\tau>n\}}X_{n+1} + 1_{\{\tau\leq n\}}Y_{n+1} \geq 1_{\{\tau>n+1\}}X_{n+1} + 1_{\{\tau\leq n+1\}}Y_{n+1}$$

so that $Z_n \geq E(Z_{n+1}|\mathcal{B}_n)$. Thus $\{Z_n,\mathcal{B}_n\}$ is a supermartingale.

□

Before we proceed to the first proposition, we need some definitions.

DEFINITION 3.3.2. Given a sequence $\{x_n\}$ in $\mathbb{R} \cup \{\infty\}$ and a pair $a < b$ of finite real numbers, we define integers τ_k $k \geq 1$ inductively by

$$\begin{aligned}
\tau_1 &= \min\{n\colon n \geq 0, & x_n \leq a\} \\
\tau_2 &= \min\{n\colon n \geq \tau_1, & x_n \geq b\} \\
\tau_3 &= \min\{n\colon n \geq \tau_2, & x_n \leq a\} \\
\tau_{2p-1} &= \min\{n\colon n \geq \tau_{2p-2}, & x_n \leq a\} \\
\tau_{2p} &= \min\{n\colon n \geq \tau_{2p-1}, & x_n \geq b\} \\
&\vdots
\end{aligned}$$

If one of the indices is not defined, for example τ_1 is not defined if $x_n > a$ for all $n \in \mathbb{N}$, then we set it to ∞. We will denote by $\beta_{a,b}$ the largest integer p for which τ_{2p} is finite, and put $\beta_{a,b} = \infty$ if all τ_k are finite.

The number $\beta_{a,b}$ represents the number of times that the sequence $\{x_n\}$ "upcrosses" the interval $[a,b]$. Thus is is clear that

$$\liminf x_n < a < b < \limsup x_n \Longrightarrow \beta_{a,b} = \infty$$

and

$$\beta_{a,b} = \infty \Longrightarrow \liminf x_n \leq a < b \leq \limsup x_n$$

from which we deduce that the sequence $\{x_n\}$ converges, possibly to ∞ or $-\infty$, if and only if $\beta_{a,b} < \infty$ for every pair $a < b$ in $\mathbb{R}$ (or, equivalently, in $\mathbb{Q}$).

Now if $\{X_n\}$ is a sequence of random variables, the indices $\tau_k(\omega)$ defined for each of the sequences $\{X_n(\omega)\}$ as above, are random variables. This is proved inductively by writing

$$\{\tau_{2p} = n\} = \sum_{m<n} \{\tau_{2p-1} = m\} \cap \{X_{m+1} < b, \dots, X_{n-1} < b,\ X_n \geq b\}$$

with an analogous formula in the case of an odd index. Since $\{\beta_{a,b} \geq p\} = \{\tau_{2p} < \infty\}$, we see that $\beta_{a,b}$ is also a random variable. We now note that

$$\{X_n(\omega) \text{ converges}\} = \bigcap_{\substack{a<b \\ a,b\in\mathbb{R}}} \{\beta_{a,b} < \infty\} = \bigcap_{\substack{a<b \\ a,b\in\mathbb{Q}}} \{\beta_{a,b} < \infty\}$$

and thus a sequence of random variables $\{X_n\}$ converges a.s. if and only if $\beta_{a,b}$ is a.s. finite for all $a < b$, $a, b \in \mathbb{Q}$.

PROPOSITION 3.3.3. If $\{X_n\}$ is a positive supermartingale then for $k \geq 1$ and $a < b \in \mathbb{R}$ we have

$$E\left(1_{\{\beta_{a,b} \geq k\}} \middle| \mathcal{B}_1\right) \leq \left(\frac{a}{b}\right)^k \min\left(\frac{X_1}{a}, 1\right)$$

Thus the random variables $\beta_{a,b}$ are a.s. finite and $\{X_n\}$ converges a.s.

PROOF ([25]): The indices τ_k are stopping times as $\{\tau_k = n\}$ depends only on $X_1, \ldots, X_n$ and thus belongs to $\mathcal{B}_n$. Thus, by repeated application of 3.3.1, we see that for a fixed $k \geq 1$, the following defines a positive supermartingale.

$$
\begin{aligned}
Y_n &= 1 && \text{if} \quad 0 \leq n < \tau_1 \\
&= X_n/a && \text{if} \quad \tau_1 \leq n < \tau_2 \\
&= \frac{b}{a} \cdot 1 && \text{if} \quad \tau_2 \leq n < \tau_3 \\
&= \frac{b}{a} \cdot \frac{X_n}{a} && \text{if} \quad \tau_3 \leq n < \tau_4 \\
&= \left(\frac{b}{a}\right)^{k-1} \cdot \frac{X_n}{a} && \text{if} \quad \tau_{2k-1} \leq n < \tau_{2k} \\
&= \left(\frac{b}{a}\right)^{k} && \text{if} \quad \tau_{2k} \leq n
\end{aligned}
$$

Note that $Y_1 = \min(1, X_1/a)$, because if X_1/a is less than 1 we have $\tau_1 = 0$, otherwise $\tau_1 > 0$ and Y_1 is 1. On the other hand, we have the inequality

$$Y_n \geq (b/a)^k 1_{\{\tau_{2k} \leq n\}}$$

Since $Y_1 \geq E(Y_n | \mathcal{B}_1)$, we find that

$$\left(\frac{b}{a}\right)^k E\left(1_{\{\tau_{2k} \leq n\}} | \mathcal{B}_1\right) \leq \min(X_1/a, 1)$$

Letting $n \uparrow \infty$ and remarking that $\{\tau_{2k} < \infty\} = \{\beta_{a,b} \geq k\}$ we get the result. Letting $k \to \infty$ and integrating both sides gives us that

$\beta_{a,b}$ is integrable and thus a.s. finite.

□

Note that if X is the a.s. limit of a positive supermartingale $\{X_n\}$, then

$$\int |X| = \int X = \lim_m \int (\inf_{n \geq m} X_m) \leq \lim_m \int X_m \leq \int X_1 < \infty$$

so that $X \in L^1$.

PROPOSITION 3.3.4. Every submartingale $\{X_n\}$ satisfying $\sup_n E(X_n^+) < \infty$ converges a.s. to a limit in L^1. In the martingale case, the preceeding condition is equivalent to the condition of L^1-boundedness.

PROOF ([25]): If $\{X_n\}$ is a submartingale, then $\{X_n^+\}$ is a positive submartingale as $E(X_{n+1}^+|\mathcal{B}_n) \geq E(X_{n+1}|\mathcal{B}_n) \geq X_n$ implies that $E(X_{n+1}^+|\mathcal{B}_n) \geq X_n^+$ for all n. For a fixed k, the sequence $\{E(X_n^+|\mathcal{B}_k),\ n \geq k\}$ is an increasing sequence of random variables as

$$E(X_{n+1}^+|\mathcal{B}_k) = E(E(X_{n+1}^+|\mathcal{B}_n)|\mathcal{B}_k) \geq E(X_n^+|\mathcal{B}_k)$$

Thus we can define

$$M_k = \lim_n \uparrow E(X_n^+|\mathcal{B}_k) \quad \text{for each } k \in \mathbb{N}$$

Clearly M_k is $\mathcal{B}_k$-measurable, and the integrability of M_k follows from hypothesis as

$$\int |M_k| = \int M_k = \lim_n \int E(X_n^+|\mathcal{B}_k) = \lim_n \int X_n^+ < \infty$$

The fact that $\{M_k\}$ is a martingale follows from 2.2.5 as

$$E(M_{k+1}|\mathcal{B}_k) = \lim_n \uparrow E(E(X_n^+|\mathcal{B}_{k+1})|\mathcal{B}_k) = \lim_n \uparrow E(X_n^+|\mathcal{B}_k) = M_k$$

Let us define $Y_n = M_n - X_n$ for each $n \in \mathbb{N}$, and note that $\{Y_n\}$ is a positive, adapted sequence in L^1. Also $\{Y_n\}$ is a supermartingale, being the difference between a martingale and a submartingale. Proposition 3.3.3 implies that the limits $M = \lim M_n$ and $Y = \lim Y_n$ exist and are in L^1. It follows that $\{X_n\}$ converges a.s. to $(M-Y)$, which is in L^1.

The relation $E(|X|) = 2E(X^+) - E(X)$ implies for a martingale the equivalence of the conditions, $\sup_n E(|X_n|) < \infty$ and $\sup_n E(X_n^+) < \infty$, $\{E(X_n)\}$ being a constant sequence.

□

PROPOSITION 3.3.5. If $\{X_n\}$ is a martingale, the following are equivalent

(i) $\{X_n\}$ converges in L^1

(ii) $\{X_n\}$ is L^1-bounded and $X = \lim X_n$, which exists by Proposition 3.3.4, satisfies $X_n = E(X|\mathcal{B}_n)$ for all n.

(iii) $\{X_n\}$ is uniformly integrable

PROOF ([25]):

(i) → (ii) Since $\{X_n\}$ converges in L^1, it is L^1-bounded so by Proposition 3.3.4, the a.s. limit X exists and must coincide with the mean limit. The continuity of the conditional expectation operator on L^1 implies that $E(X_k|\mathcal{B}_n) \to E(X|\mathcal{B}_n)$ in L^1 as $k \to \infty$, for any $n \in N$. But if $k \geq n$, $E(X_k|\mathcal{B}_n) = X_n$ so $X_n = E(X|\mathcal{B}_n)$ for all n.

(ii) → (iii) By 2.3.5, $\{E(X|\mathcal{B}_n), n \in N\}$ is uniformly integrable.

(iii) $\rightarrow$ (i) By uniform integrability, X_n is L^1-bounded and thus has an a.s. limit X. By 2.3.6, $\{X_n\}$ converges to X in L^1. $\square$

Now that we have considered martingale convergence results let us turn to martingale-like sequences. In particular, we are interested in how the hypotheses of positivity, L^1-boundedness, and uniform integrability affect the convergence of martingale-like sequences.

These results have been condensed into the following table. A check (✓) indicates that the given condition is sufficient for a given type of convergence, and a cross (×) indicates that it is not. The comments, which follow the table, show where proofs and counter-examples can be found to justify the table entries.

TABLE 3.3.6

Process	Positive	L^1-bounded	Uniformly	Integrable
	a.s.	a.s.	a.s.	L^1
GFT	×	×	×	✓ (i)
L^1-martingale	× (ii)	×	× (ii)	✓
MIL	(iii)	✓ (iv)	✓	✓
Amart	✓ (v)	✓	✓	✓
Progressive	✓	✓	✓	✓
Quasimartingale	✓	✓	✓	✓
Martingale	✓	✓	✓	✓
Eventual	✓ (vi)	×	×	× (vii)

(i) Proposition 3.3.7 states that a uniformly integrable GFT converges in L^1. Referring to Diagram 3.2.9 we see that the conclusion must hold for all of the others, except eventual martingales.

(ii) The process in Example 3.2.18 is a uniformly integrable, positive L^1-martingale (and thus also a GFT) which doesn't converge almost surely.

(iii) The question of whether or not a positive MIL converges a.s. or not is still open. Some partial results are given in Propositions 3.3.12 and 3.3.14.

(iv) Proposition 3.3.8 states that an L^1-bounded MIL converges a.s. . Referring to Diagram 3.2.9 we see that the conclusion must hold for all the others, excepting eventual martingales, L^1-martingales and GFTs.

(v) Proposition 3.4.5 states that a positive amart must converge a.s. . By Diagram 3.2.9 this is also true for martingales and quasimartingales.

(vi) By Corollary 3.3.10, a positive eventual martingale converges a.s., and thus, so must a progressive martingale.

(vii) Proposition 3.3.13 gives an example of a uniformly integrable eventual martingale which does not converge in L^1 nor almost surely.

PROPOSITION 3.3.7. Every uniformly integrable GFT converges in L^1.

Note: Mucci [22] gave a proof for the above proposition which appeared in the same volume of the same journal as the proof given below.

PROOF ([30]): Let $\{X_n\}$ be a uniformly integrable GFT. We let Γ denote the family $\{E(X_n|\mathcal{B}_m)|m \in N \;\; n \geq m\}$. The proof is broken into five parts to make it easier to follow.

(i) Γ is uniformly integrable.

Since $\{X_n\}$ is uniformly integrable we can find G as in Proposition 2.3.4. Then for $n \geq m$

$$\begin{aligned} E(G \circ |E(X_n|\mathcal{B}_m)|) \leq E(G \circ E(|X_n||\mathcal{B}_m)) &\leq E(E(G \circ |X_n||\mathcal{B}_m)) \\ &= E(G \circ |X_n|) \end{aligned}$$

using 2.2.6 and the fact that G is non-decreasing and convex, and the fact that the absolute value function is convex. Thus

$$\sup_{\Gamma} E(G \circ |E(X_n|\mathcal{B}_m)|) \leq \sup_{n} E(G \circ |X_n|) < \infty$$

so by a further application of Proposition 2.3.4 we see that Γ is uniformly integrable.

(ii) Given $\varepsilon > 0$, there exists M such that $m \geq M$ implies

$$E(|E(X_n-X_m|\mathcal{B}_m)|) \leq 2\varepsilon \quad \text{for any} \quad n \geq m.$$

Let $\varepsilon > 0$. Since Γ is uniformly integrable, by 2.3.2, we can find $\delta > 0$ such that $P(B) < \delta \Rightarrow \int_B |E(X_n|\mathcal{B}_m)| < \varepsilon$ for all $m \in N \;\; n \geq m$.

Since $\{X_n\}$ is a GFT we can choose M so large that $n \geq m \geq M$ implies $P[|E(X_n-X_m|\mathcal{B}_m)| > \varepsilon] < \delta$. Then

$$\int |E(X_n - X_m | \mathcal{B}_m)| \leq \varepsilon + \int_{\{|E(X_n - X_m|\mathcal{B}_m)| > \varepsilon\}} |E(X_n - X_m | \mathcal{B}_m)| \leq 2\varepsilon$$

(iii) For m fixed, the sequence $\{E(X_n|\mathcal{B}_m)\}$ converges in L^1 to a $\mathcal{B}_m$-measurable random variable Z_m.

Let $m \leq n \leq n'$.

$$\begin{aligned} E(|E(X_{n'}|\mathcal{B}_m) - E(X_n|\mathcal{B}_m)|) &= E(|E(X_{n'} - X_n|\mathcal{B}_m)|) \\ &= E(|E(E(X_{n'} - X_n|\mathcal{B}_n)|\mathcal{B}_m)|) \\ &\leq E(E(|E(X_{n'} - X_n|\mathcal{B}_n)||\mathcal{B}_m)) \\ &= E(|E(X_{n'} - X_n|\mathcal{B}_n)|) = E(|E(X_{n'}|\mathcal{B}_n) - X_n|) \end{aligned}$$

which by (ii) can be made as small as required. Thus for fixed m, $\{E(X_n|\mathcal{B}_m)\}$ is Cauchy and hence converges in L^1, to say Z_m. Since there is a subsequence of $\{E(X_n|\mathcal{B}_m)\}$ converging a.s. to Z_m, it is without loss of generality that we assume Z_m to be $\mathcal{B}_m$-measurable.

(iv) $\{Z_m, \mathcal{B}_m\}$ is a uniformly integrable martingale.

By 2.3.3 we get that $\{Z_m\}$ is uniformly integrable. By the continuity of the conditional expectation operator on L^1 we see that $E(E(X_n|\mathcal{B}_{m+1})|\mathcal{B}_m) \to E(Z_{m+1}|\mathcal{B}_m)$ in L^1. Thus there is a subsequence n' of $\{n: n \geq m\}$ such that $E(E(X_{n'}|\mathcal{B}_{m+1})|\mathcal{B}_m) \to E(Z_{m+1}|\mathcal{B}_m)$ a.s.

By choosing another subsequence if necessary we can assume that $E(X_{n'}|\mathcal{B}_m) \to Z_m$ a.s., so that

$$\begin{aligned} E(Z_{m+1}|\mathcal{B}_m) &= \lim_{n'} E(E(X_{n'}|\mathcal{B}_{m+1})|\mathcal{B}_m) \text{ a.s.} \\ &= \lim_{n'} E(X_{n'}|\mathcal{B}_m) \text{ a.s.} = Z_m \text{ a.s.} \end{aligned}$$

so that $\{Z_m\}$ is a martingale.

Finally we arrive at the proof of the proposition

(v) $\{X_n\}$ converges in L^1.

Since $\{Z_n\}$ is a uniformly integrable martingale, by 3.3.5 we can find $Z \in L^1$ such that $Z_n \to Z$ in L^1. Let $\varepsilon > 0$. From (ii) and (iii) we see that we can find M such that $m \geq M$ impleis

$$\int |X_m - Z_m| \leq 2\varepsilon$$

Thus, for sufficiently large m

$$\int |X_m - Z| \leq \int |X_m - Z_m| + \int |Z_m - Z| \leq 3\varepsilon$$

and so $\{X_n\}$ converges in L^1 to Z.

□

PROPOSITION 3.3.8. Every L^1-bounded MIL converges a.s.

PROOF ([23]): The proof of this proposition is similar to the proof of Proposition 3.3.3 which says that a positive supermartingale converges almost surely. Again we proceed by showing that the number of times that the process $\{X_n\}$ crosses the interval $[a,b]$ is almost surely finite, for any $a, b \in \mathbb{R}$. This will give us an a.s. limit X and since

$$\int |X| \leq \liminf \int |X_n| < \infty$$

we see that $X \in L^1$.

Let $\{X_n\}$ be an L^1-bounded MIL. We begin by defining a sequence of stopping times. Let $\tau_o = 0$, and let $\{\alpha_n\}$ be a

decreasing sequence of positive numbers with $\Sigma\alpha_n < \infty$. Let N be a positive integer and let $a < b \in \mathbb{R}$. For $n \in \mathbb{N}$, define τ_{2n-1} as the first $m \leq N$ such that

(i) $m > \tau_{2n-2}$

(ii) $X_m > b$

(iii) $\sup_{k \geq m} |E(X_k - X_m | \mathcal{B}_m)| < \alpha_n$

If no such m exists, let $\tau_{2n-1} = N$.

Similarly, define τ_{2n} as the first $m \leq N$ such that

(iv) $m > \tau_{2n-1}$

(v) $X_m < a$

(vi) $\sup_{k \geq m} |E(X_k - X_m | \mathcal{B}_m)| < \alpha_n$

If no such m exists, let $\tau_{2n} = N$. Now we have

$$\begin{aligned}\int X_{\tau_{2n-1}} - \int X_{\tau_{2n}} &= \sum_1^N \int_{\{\tau_{2n-1}=k\}} (X_k - E(X_N | \mathcal{B}_k)) + \sum_1^N \int_{\{\tau_{2n}=k\}} (E(X_N | \mathcal{B}_k) - X_k) \\ &< 2\alpha_n .\end{aligned}$$

Thus

$$\sum_1^\infty \int \left(X_{\tau_{2n-1}} - X_{\tau_{2n}}\right) < 2\sum \alpha_n < \infty$$

Now define

$$\bar{\beta}_{a,b}(N) = \sum_{1}^{\infty} \left(1_{\left\{X_{\tau_{2n-1}} \geq b\right\}} 1_{\left\{X_{\tau_{2n}} \leq a\right\}} 1_{\left\{\sup\limits_{m \geq 0} \left| E\left(X_{\tau_{2n+m}} - X_{\tau_{2n}} \middle| \mathcal{B}_{\tau_{2n}}\right) \right| < \alpha_n\right\}} \right)$$

Then $\bar{\beta}_{a,b}(N)$ is the number of "downcrossings" that the finite process $\{X_0,\ldots,X_n\}$ makes over the interval $[a,b]$, subject to the conditions (iii) and (vi). We have $\sum(X_{\tau_{2n-1}}-X_{\tau_{2n}}) \geq (b-a)\bar{\beta}_{a,b}(N) - |b| - |X_N|$.

Define $\bar{\beta}_{a,b} = \liminf_N \bar{\beta}_{a,b}(N)$, taking integrals and using Fatou's lemma we get

$$\int \bar{\beta}_{a,b} < \frac{1}{b-a} \; [|b| + 2\alpha + \sup_n \int |X_n|] < \infty$$

Therefore $P[\bar{\beta}_{a,b} < \infty] = 1$.

Now define $\Omega_0 = \{\bar{\beta}_{a,b} < \infty\} \cap \left\{\lim_n \sup_{k \geq n} |E\{X_k| \ _n\}-X_n| = 0\right\}$

Clearly $P(\Omega_0) = 1$. Let $\beta_{a,b}$ denote the actual number of downcrossings of the process $\{X_n\}$ over $[a,b]$. If $M \in \mathbb{N}$ and we let $A = \{\bar{\beta}_{a,b} = M\} \cap \{\beta_{a,b} = \infty\} \cap \Omega_0$, then on A we can find a sequence $\{n_k\}$ where $X_{n_{2k-1}} \geq b$ and $X_{n_{2k}} \leq a$ and (iii) and (vi) hold, which contradicts $\bar{\beta}_{a,b} = M$. Thus $A = \phi$ and $P[\beta_{a,b} = \infty] = 0$ so $\{X_n\}$ converges a.s.

□

We now turn our attention to the convergence properties of eventual martingales. When Tomkins introduced this type of process he gave two decompositions of a given eventual martingale $\{X_n\}$ into the sum of a martingale and a process which is a.s. convergent, i.e. $X_n = M_n + Z_n$. However, since the martingale $\{M_n\}$ doesn't necessarily inherit the boundedness properties of the original process, his conditions had to be very stringent in order to get almost sure convergence.

We now show that a "predictably positive" eventual supermartingale converges a.s., which gives as a corollary that a positive eventual supermartingale converges a.s.

PROPOSITION 3.3.9. Suppose $\{X_n\}$ is an eventual supermartingale, and suppose that there is a sequence of events $\{A_n\}$ such that $P[\liminf A_n] = 1$, and $A_n \in \mathcal{B}_{n-1}$ and $X_n \geq 0$ on A_n for each $n \in \mathbb{N}$. Then $\{X_n\}$ converges a.s.

PROOF: It suffices for each $\varepsilon > 0$ to find a subset $\Omega_\varepsilon \subseteq \Omega$ such that $P[\Omega_\varepsilon] > 1 - \varepsilon$ and $\{X_n(\omega)\}$ converges to a finite limit for almost all $\omega \in \Omega_\varepsilon$, for then

$$P[X_n \text{ converges}] = P[\cup\Omega_{1/n}] = 1$$

Let $\varepsilon > 0$. Since $\{X_n\}$ is an eventual supermartingale and $P[\liminf A_n] = 1$, we can find N such that $P[C_N] > 1 - \varepsilon$ where

$$C_N = \bigcap_{k=N}^{\infty} \left(\{E(X_{k+1}-X_k|\mathcal{B}_k) \leq 0\} \cap A_{k+1}\right)$$

For $j \geq N$ define $B_j = \bigcap_{k=N}^{j} \left(\{E(X_{k+1}-X_k|\mathcal{B}_k) \leq 0\} \cap A_{k+1}\right)$. Then $B_j \in \mathcal{B}_j$ and $\lim_j \downarrow B_j = C_N$. Now define $Y_j = X_j 1_{B_j}$ for $j \geq N$. Then

$$\begin{aligned} E(Y_{j+1}|\mathcal{B}_j) &= E\left(X_{j+1}1_{B_{j+1}}|\mathcal{B}_j\right) \\ &\leq E\left(X_{j+1}1_{B_j}|\mathcal{B}_j\right) \quad \text{since } X_{j+1} \geq 0 \text{ on } B_j \\ &= E(X_{j+1}|\mathcal{B}_j)1_{B_j} \leq X_j 1_{B_j} = Y_j \end{aligned}$$

Thus $\{Y_j\}$ is a supermartingale, and since $X_j \geq 0$ on $A_j \supseteq B_j$, it is positive, so by 3.3.3 $\{Y_j\}$ converges almost surely. The proof is completed by noting that $X_j = Y_j$ on C_N.

□

COROLLARY 3.3.10. Every positive eventual supermartingale converges almost surely.

PROOF: Let $A_n = \Omega$ for all $n \in \mathbb{N}$ and apply 3.3.9.

□

Note that unlike the corresponding limit for positive supermartingales, if $\{X_n\}$ is a positive eventual supermartingale, we cannot conclude that $\lim X_n \in L^1$. The process described in 3.2.19 is a counterexample.

The form of Proposition 3.3.9 suggests a reason why when every positive eventual martingale converges a.s., not every eventually positive eventual martingale converges a.s. The reason is that the set $\{X_n \geq 0\}$ does not necessarily belong to $\mathcal{B}_{n-1}$ for each $n \in \mathbb{N}$. An example of an eventually positive eventual martingale which does not converge a.s. can be made by adding the constant 1 to the eventual martingale described in Proposition 3.3.13.

COROLLARY 3.3.11. Let $\{X_n\}$ be an eventual martingale such that either $\sup X_n$ or $\inf X_n$ is in L^1. Then $\{X_n\}$ converges almost surely.

PROOF: Suppose $\inf X_k \in L^1$. Write $X_n = Y_n + Z_n$ where $Y_n = X_n - E(\inf X_k|\mathcal{B}_n)$ and $Z_n = E(\inf X_k|\mathcal{B}_n)$. Clearly $\{Y_n\}$ and $\{Z_n\}$ are adapted. Now

$$X_n - \inf X_k \geq 0 \Rightarrow Y_n = E(X_n - \inf X_k|\mathcal{B}_n) \geq 0$$

Also, on the set $\{E(X_n - X_{n-1}|\mathcal{B}_{n-1}) = 0\}$ we have

$$\begin{aligned} E(Y_n|\mathcal{B}_{n-1}) &= E(X_n|\mathcal{B}_{n-1}) - E(E(\inf X_k|\mathcal{B}_n)|\mathcal{B}_{n-1}) \\ &= X_{n-1} - E(\inf X_k|\mathcal{B}_{n-1}) \\ &= Y_{n-1} \end{aligned}$$

so that $\{Y_n\}$ is a positive eventual martingale, which by 3.3.9 converges a.s.

Furthermore $E(Z_{n+1}|\mathcal{B}_n) = E(E(\inf X_k|\mathcal{B}_{n+1})|\mathcal{B}_n) = E(\inf X_k|\mathcal{B}_n) = Z_n$ so $\{Z_n\}$ is a martingale which is L^1-bounded and hence a.s. convergent. Thus $\{X_n\}$ converges a.s.

In case $\sup X_n \in L^1$, apply the above argument to the eventual martingale $\{-X_n\}$.

□

COROLLARY 3.3.12. Every positive MIL $\{X_n\}$ converges in probability.

PROOF: By a result in real analysis, $X_n \to X$ in probability is equivalent to having every subsequence produce a further subsequence which converges a.s. to X. Since a subsequence of a positive MIL is a positive MIL, it suffices to show that if $\{X_n\}$ is a positive MIL, then there exists a subsequence $\{X_{n_k}\}$ converging almost surely.

Let $\{X_n\}$ be a positive MIL and let $\{a_k\}$ be a decreasing sequence of positive numbers such that $\Sigma a_k < \infty$. Since $\{X_n\}$ is MIL we can find $\{n_k\}$, an increasing sequence of integers such that for each $k \in \mathbb{N}$

$$P\left[\left(\sup_{n \geq n_k} \left|E\left(X_n - X_{n_k}|\mathcal{B}_{n_k}\right)\right|\right) \geq a_k\right] < a_k$$

For each $k \in \mathbb{N}$, let $Y_k = X_{n_k} + \sum_{i=k}^{\infty} a_i$. Then

$$E\left(Y_{k+1}-Y_k \mid \mathcal{B}_{n_k}\right) = E\left(X_{n_{k+1}}-X_{n_k} \mid \mathcal{B}_{n_k}\right) - a_k$$

Since $P[E(Y_{k+1}-Y_k \mid \mathcal{B}_{n_k}) \leq 0] > 1 - a_k$, by applying the Borel-Cantelli lemma we see that $\{Y_k, \mathcal{B}_{n_k}\}$ is a positive eventual supermartingale. By 3.3.10 $\{Y_k\}$ converges a.s., and since $\Sigma a_k < \infty$ we see that $\{X_{n_k}\}$ converges a.s. so the proposition is proved.

□

The process constructed in the following proposition demonstrates that eventual martingales do not share the convergence properties of GFT or MIL. In particular, an eventual martingale need not converge in any reasonable sense, even if it is uniformly integrable.

PROPOSITION 3.3.13. For $1 \leq p < \infty$, there exists a uniformly integrable, L^p-bounded eventual martingale which does not converge weakly in L^1. This implies that it converges neither a.s. nor in L^1.

PROOF: It is without loss of generality that we suppose $p > 1$, for an L^p-bounded process is also L^1-bounded.

Let $n \in \mathbb{N}$ and $0 < q < 1$. Define the finite sequence $S(n,q)$ of independent random variables as follows

$$S(n,q) = \{X_0, X_1, \ldots, X_{2n}, X_{2n+1}\}$$

where $X_0 = 1$, $P[X_1 = 1 - \frac{1}{n}] = 1 - q$ and $P[X_1 = 1 + \frac{1}{nq} + \frac{1}{n}] = q$. For $k = 2,\ldots,2n$, X_k is distributed as $X_1 - (k-1)/n$ and finally $X_{2n+1} = -1$.

We define $-S(n,q)$ as $\{-X_0, -X_1, \ldots, -X_{2n+1}\}$.

If we use the filtration $\mathcal{B}_k = \sigma(X_j, j \leq k)$ we observe that $P[E(X_1 - X_0 | \mathcal{B}_0) \neq 0] = P[X_0 \neq 1] = 0$ since $E(X_1) = 1$. Also $P[E(X_2 - X_1 | \mathcal{B}_1) \neq 0] = P[X_1 \neq 1 - \frac{1}{n}] = q$ and similarly for $2 \leq k \leq 2n$, $P[E(X_{k+1} - X_k | \mathcal{B}_k) \neq 0] = q$. Thus $P[E(X_{k+1} - X_k | \mathcal{B}_k) \neq 0$ for some X_{k+1}, $X_k \in S(n,q)] \leq 2nq$.

Now let $c \in \mathbb{N}$ and define for $k \geq 2$, $n_k = k^c$, $q_k = 1/k^{c+2}$. Let $\{X_n\}$ be a sequence of independent random variables whose marginal distributions are the same as the marginal distributions of the sequence $\{S(n_2,q_2), -S(n_2,q_2), S(n_3,q_3), -S(n_3,q_3)\ldots\}$ and $\mathcal{B}_n = \sigma(X_k, k \leq n)$. Then $\{X_n, \mathcal{B}_n\}$ is the required process. Note that the final random variable in any of the finite sequences is the same as the first random variable in the following finite sequence, namely either 1 or -1. Therefore

$$\sum_{n=1}^{\infty} P[E(X_{n+1} - X_n | \mathcal{B}_n) \neq 0] \leq 2 \cdot \sum_k 2/k^2 < \infty,$$

so by the Borel-Cantelli lemma, $\{X_n\}$ is an eventual martingale.

Also for $n \in \mathbb{N}$ we have

$$\int |X_n|^p \leq \int_{\{|X_n|>1\}} |X_n|^p + 1 \leq \left\{1 + \frac{1}{n_k q_k} + \frac{1}{n_k}\right\}^p \cdot q_k + 1$$

$$\leq \left(\frac{3}{n_k q_k}\right)^p \cdot q_k + 1 = \frac{3^p k^{2p}}{k^{c+2}} + 1$$

for some $k \in \mathbb{N}$. Therefore $\{X_n\}$ is L^p-bounded if we choose $c+2 \geq 2p$. Furthermore, by applying 2.3.4 with $G(x) = x^p$ we see that $\{X_n\}$ is uniformly integrable.

Since $\limsup E(X_n) = 1$ and $\liminf E(X_n) = -1$, $\{E(X_n)\}$ does not converge and so $\{X_n\}$ does not converge weakly in L^1. The

last statement of the proposition follows since L^1 convergence implies weak convergence, and under uniform integrability a.s. convergence implies L^1 convergence.

□

We conclude this section with a partial solution to the question: Does a positive MIL always converge a.s.?

PROPOSITION 3.3.14. Let $\{X_n\}$ be a positive MIL. Either of the following conditions is sufficient to imply almost sure convergence.

(i) $\{X_n\}$ is an L^1-martingale.

(ii) For each $m \in N$ $\lim_{n \geq m} E(X_n | \mathcal{B}_m)$ exists and is finite a.s. and $\sup_{n \geq m} E(X_n | \mathcal{B}_m) \in L^1$.

PROOF: In [26] Peligrad shows that if $\{Y_n\}$ is a MIL and an L^1-martingale, and also $\sup_n \int Y_n^+ < \infty$, then $\{Y_n\}$ converges a.s. Since the process $\{-X_n\}$ satisfies these three conditions it must converge a.s., and so must $\{X_n\}$.

In [5], Blake shows that the conditions in (ii) imply, for a MIL, that $\{X_n\}$ has a weak Riesz decomposition (see 3.4.1) with $Z_n \to 0$ a.s. . The rest of the proof is as in 3.4.5.

□

3.4 Weak Riesz Decomposition

The weak Riesz decomposition of a stochastic process is a generalization of the Riesz decomposition of a positive supermartingale into the sum of a martingale and a potential. (A potential is a positive supermartingale with $E(X_n) \to 0$). We begin this section by

characterizing processes with weak Riesz decompositions, and then show which of the martingale-like processes have one. Finally, this decomposition is used to prove an amart convergence result.

DEFINITION 3.4.1. A process $\{X_n, \mathcal{B}_n\}$ is said to have a weak Riesz decomposition [abbreviated WRD] if it can be written $X_n = Y_n + Z_n$ $n \in \mathbb{N}$, where $\{Y_n, \mathcal{B}_n\}$ is a martingale and $E(Z_n \cdot 1_A) \to 0$ for any $A \in \cup\mathcal{B}_n$. This decomposition is unique.

PROPOSITION 3.4.2. $\{X_n\}$ has a WRD if and only if $E(X_n \cdot 1_A)$ converges to a finite limit for any $A \in \cup\mathcal{B}_n$.

PROOF: ($\Rightarrow$) Let $\{X_n\}$ have a WRD and let $A \in \mathcal{B}_m$. Using the fact that $\{Y_n\}$ is a martingale, for $n \geq m$

$$\int_A X_n = \int_A Y_n + \int_A Z_n = \int_A Y_m + \int_A Z_n$$

By hypothesis, this converges to $\int_A Y_m < \infty$ as $n \to \infty$.

($\Leftarrow$) Suppose $E(X_n \cdot 1_A)$ converges for all $A \in \cup\mathcal{B}_n$.

Let $m \in \mathbb{N}$ and define $\nu_n(A) = \int_A X_n$ on $\mathcal{B}_m$. Define $\nu(A) = \lim_{n \geq m} \nu_n(A)$ on $\mathcal{B}_m$, by assumption this limit exists. Now by a corollary to the Vitali-Hahn-Saks theorem [12], ν is a signed measure on $\mathcal{B}_m$.

Since $\nu_n << P$ on $\mathcal{B}_m$ for all n, it follows that $\nu << P$ on $\mathcal{B}_m$ and so by the Radon-Nikodym theorem, there exists a $\mathcal{B}_m$-measurable random variable Y_m such that

$$\nu(A) = \int_A Y_m \quad \text{on} \quad \mathcal{B}_m.$$

We also note that $Y_m \in L^1$ as

$$\int |Y_m| = \int Y_m^+ - \int Y_m^- = \nu(\{Y_m \geq 0\}) - \nu(\{Y_m < 0\}) < \infty$$

Using the above construction for each $m \in \mathbb{N}$ we get an adapted sequence $\{Y_m\}$ and we note that if $A \in \mathcal{B}_m$ and $n \geq m$ then

$$\int_A Y_m = \lim_{k \geq m} \nu_k(A) = \lim_{k \geq n} \nu_k(A) = \int_A Y_n$$

so that $E(Y_n | \mathcal{B}_m) = Y_m$ and $\{Y_m\}$ is a martingale. Furthermore for $A \in \mathcal{B}_m$

$$\int_A Z_n = \int_A X_n - Y_n = \nu_n(A) - \nu(A) \to 0 \quad \text{as} \quad n \to \infty$$

so $\{X_n\}$ has a WRD.

$\square$

PROPOSITION 3.4.3. The following are equivalent

(i) $\{X_n\}$ is uniformly integrable and has a WRD.

(ii) $\{X_n\}$ is L^1-bounded and $E(X_n 1_B)$ converges for each $B \in \sigma(\cup \mathcal{B}_n) = \mathcal{B}$.

(iii) $\{X_n\}$ converges weakly in L^1 to some $X \in L^1$.

PROOF: We use the metric space of P-equivalent sets in $\mathcal{B}$, denoted $\mathcal{B}/P$, with the metric $d(B,B') = P(B \Delta B')$. We also recall that the closure of an algebra in this space is a σ-algebra.

(i) $\to$ (ii) Let $B \in \mathcal{B}$ and for $\delta > 0$ let $B' \in \cup \mathcal{B}_n$ such that $P(B \Delta B') < \delta$. Thus for any n, m

$$\left|\int_B X_n - \int_B X_m\right| \leq \left|\int_{B'} X_n - \int_{B'} X_m\right| + \int_{B \Delta B'} |X_n| + \int_{B \Delta B'} |X_m|$$

By uniform integrability the two right hand terms can be made uniformly small in n,m by choosing δ small enough. Since $\{X_n\}$ has a WRD, by 3.4.2 the first term on the right hand side converges to zero. Thus $E(X_n 1_B)$ converges to a finite limit for all $B \in \mathcal{B}$, since $\{E(X_n 1_B)\}$ is Cauchy.

(ii) $\rightarrow$ (i) ([24]) Clearly, by 3.4.2, $\{X_n\}$ has a WRD. Since $\{X_n\}$ is L^1-bounded, to show uniform integrability we only need to show that for $\varepsilon > 0$ there exists $\delta > 0$ such that $P(B) < \delta$ implies $\sup_n \int_B |X_n| \leq \varepsilon$.

Let $\varepsilon > 0$. Since the map $B \rightarrow E(X1_B)$ of the metric space $\mathcal{B}/P$ into $\mathbb{R}$ is continuous for any random variable X, the set $\{B: |E(X1_B)| \leq \varepsilon\}$ is closed for each $\varepsilon > 0$. By hypothesis, the union of the closed sets

$$F_N = \bigcap_{\substack{m \geq N \\ n \geq N}} \{B: |\int_B (X_n - X_m)| \leq \varepsilon\}$$

is the whole space.

The Baire category theorem now implies the existence of an $N_0 \in \mathbb{N}$ such that F_{N_0} has an interior point. That is, there exists $N_0 \in \mathbb{N}$, $B_0 \in \mathcal{B}/P$ and $r \in \mathbb{R}$ such that

$$|\int_B (X_n - X_m)| \leq \varepsilon \quad \text{if} \quad m,n \geq N_0 \quad \text{and} \quad P(B \,\Delta\, B_0) \leq r$$

Let $B \in \mathcal{B}/P$ such that $P(B) \leq r$. Then since $(B_0 \cup B) \,\Delta\, B_0 \subset B$ and $(B_0 \cap B^c) \,\Delta\, B_0 \subset B$ and

$$\int_B X = \int_{B_0 \cup B} X - \int_{B_0 \cap B^c} X$$

we get $|\int_B (X_m - X_n)| \leq 2\varepsilon$ if $m,n \geq N_0$.

Applying this inequality separately on $B \cap \{X_m \leq X_n\}$ and $B \cap \{X_m > X_n\}$ we see that

$$\int_B |X_m - X_n| \leq 4\varepsilon.$$

Since the finite sequence $\{X_n\}$ $n = 1,\ldots,N_0$ is uniformly integrable, it is easy to find $\delta > 0$ such that if $P(B) < \delta$ then

$$\sup_n \int_B |X_n| \leq 5\varepsilon.$$

Thus $\{X_n\}$ is uniformly integrable so (i) holds.

The equivalence of (ii) and (iii) is a result in functional analysis. The interested reader is referred to Theorem 7, on page 291 of [12].

□

PROPOSITION 3.4.4. Every L^1-martingale $\{X_n\}$ has a WRD, and furthermore $\{Z_n\}$ converges to zero in L^1.

PROOF: For $n > m > k$ and $A \in \mathcal{B}_k$

$$|\int_A X_n - \int_A X_m| = |\int_A E(X_n - X_m|\mathcal{B}_m)| \leq \|E(X_n - X_m|\mathcal{B}_m)\|_1$$

By hypothesis the right hand side converges to zero, thus so must the left hand side. By 3.4.2 $\{X_n\}$ has a WRD.

To prove that $\{Z_n\}$ converges to zero in L^1 we begin by showing that $\{Z_n\}$ is uniformly integrable. Since $E(Z_n - Z_m|\mathcal{B}_m) = E(X_n - X_m|\mathcal{B}_m)$ for $n \geq m$ we see that $\{Z_n\}$ is an L^1-martingale. Let $\varepsilon > 0$ and choose M such that $n \geq m \geq M \Rightarrow \|E(Z_n - Z_m|\mathcal{B}_m)\|_1 < \varepsilon/3$.

Also choose k such that

$$\sup_{1\le i\le M} \int_{\{|Z_i|>k\}} |Z_i| < \varepsilon$$

Now let $m \ge M$ and let $B = \{Z_m > k\}$ and $A = \{Z_m < -k\}$. Then

$$\int_{\{|Z_m|>k\}} |Z_m| = \int_B Z_m - \int_A Z_m, \text{ and for } n \ge m$$

$$= |\int_B Z_n - \int_B (Z_n - Z_m) - \int_A Z_n + \int_A (Z_n - Z_m)|$$

$$\le |\int_B Z_n| + |\int_A Z_n| + |\int_B E(Z_n - Z_m|\mathcal{B}_m)| + |\int_A E(Z_n - Z_m|\mathcal{B}_m)|$$

$$\le |\int_B Z_n| + |\int_A Z_n| + \|E(Z_n - Z_m|\mathcal{B}_m)\|_1$$

Since each of the terms on the right converges to zero, we can choose n so that they are less than $\varepsilon/3$ and, we have

$$\sup_{1\le i<\infty} \int_{\{|Z_i|>k\}} |Z_i| < \varepsilon$$

so $\{Z_n\}$ is uniformly integrable.

Since $\{Z_n\}$ is a uniformly integrable L^1-martingale, by 3.3.7 it converges in L^1, to Z say. For $k \in \mathbb{N}$ and $A \in \mathcal{B}_k$ we have

$$\int_A Z = \lim_n \int_A Z_n = 0$$

so $E(Z|\mathcal{B}_k) = 0$ for $k \in \mathbb{N}$. Since $\{E(Z|\mathcal{B}_k)\}\ k \in \mathbb{N}$ is a martingale which converges to Z in L^1, we see that $Z = 0$.

□

PROPOSITION 3.4.5. Every positive amart converges a.s.

PROOF: Let $X_n = Y_n + Z_n$ be the WRD of $\{X_n\}$. Since $\{Y_n\}$ is a martingale, if $\tau \in T$ and $\tau \leq M$ then

$$\int Y_\tau = \sum_k \int_{\{\tau=k\}} Y_k = \sum_k \int_{\{\tau=k\}} Y_M = E(Y_M) = E(Y_1)$$

So $\int X_\tau = \int Y_\tau + \int Z_\tau$, which tells us that $\{Z_n\}$ is an amart.

By 3.4.4 $\{Z_n\}$ converges to zero in L^1 so it is L^1-bounded. Since $\{Z_n\}$ is a MIL, by 3.3.8 it converges a.s. to zero.

Since $X_n \geq 0$ for each $n \in \mathbb{N}$, referring to the construction of the WRD in 3.4.2, we see that $\nu_n(A) \geq 0$ for each n which implies $\nu(A) \geq 0$ for all $A \in \mathcal{B}_k$. Thus $Y_k \geq 0$ so $\{Y_k\}$ is a positive martingale. Thus $\{Y_k\}$ converges a.s. and so must $\{X_n\}$.

□

We note that the progressive martingale described in 3.2.19 does not have a WRD since $E(X_n)$ diverges to infinity. We have shown that an L^1-martingale has a WRD, so by considering Diagram 3.2.9, we see that we have a complete picture of the relationship between the WRD and the martingale-like processes.

3.5 Sampling, Stopping and Transforms

In this section we try to extend three important martingale results to martingale-like processes. In the table below, an asterik (*) stands for a type of process, while a check (✓) indicates that the theorem holds for the corresponding process and a cross (×) indicates that it does not. We note that the stopping theorems are special cases of the sampling theorems.

THEOREM 3.5.1. Optional Sampling Theorem [abbreviated OSAT]

If $\{X_n\}$ is a $(*)$ on $(\Omega,\mathcal{B},P,\{\mathcal{B}_n\})$ and if $\{\tau_k\}$ is a non-decreasing sequence in T, then $\{X_{\tau_k}\}$ is or is not a $(*)$ on $(\Omega,\mathcal{B},P,\{\mathcal{B}_{\tau_k}\})$ as given in Table 3.5.4.

THEOREM 3.5.2. Optimal Stopping Theorem [abbreviated OSTT]

If $\{X_n\}$ is a $(*)$ on $(\Omega,\mathcal{B},P,\{\mathcal{B}_n\})$ and τ is a stopping time, then $\{X_{n\wedge\tau}\}$ is or is not a $(*)$ on $(\Omega,\mathcal{B},P,\{\mathcal{B}_{n\wedge\tau}\})$ as given in Table 3.5.4. $(n\wedge\tau = \min\{n,\tau\}$ and $n\vee\tau = \max\{n,\tau\})$.

THEOREM 3.5.3. Transform Theorem [abbreviated TT]

If $\{X_n\}$ $n \in \mathbb{N}_+$ is a $(*)$ on $(\Omega,\mathcal{B},P,\{\mathcal{B}_n\})$, U_0 is $\mathcal{B}_0$-measurable and $\{U_n\}$ $n \in N$ is a sequence of random variables such that U_n is $\mathcal{B}_{n-1}$-measurable and $U_n \in L^\infty$ for each $n \in \mathbb{N}$, then the sequence $(U*X)_n$, called the *transform* of $\{X_n\}$, defined as follows $(U*X)_0 = U_0X_0$, $(U*X)_{n+1} - (U*X)_n = U_{n+1}(X_{n+1}-X_n)$, $n \in \mathbb{N}_+$ is or is not a $(*)$ on $(\Omega,\mathcal{B},P,\{\mathcal{B}_n\})$ as given in Table 3.5.4.

The remainder of this section is comprised of proofs and examples justifying the entries in Table 3.5.4. We begin with the results on sampling and stopping.

TABLE 3.5.4

*	OSAT	OSTT	TT
Martingale	✓	✓	✓
Sub or super martingale	✓	✓	×
Quasimartingale	✓	✓	×
Amart	✓	✓	×
Progressive	×	✓	×
Eventual	×	✓	✓
GFT	×	×	×
MIL	×	×	×
L^1-martingale	×	×	×

Sampling and Stopping

We need a few lemmas for the proofs which follow.

LEMMA 3.5.5. If $\nu_1 \leq \nu_2$ are two stopping times then $\mathcal{B}_{\nu_1} \subseteq \mathcal{B}_{\nu_2}$.

PROOF: Let $A \in \mathcal{B}_{\nu_1}$, then $A \cap \{\nu_1 = n\} \in \mathcal{B}_n$ for $n \in \mathbb{N}$, thus $A \cap \{\nu_1 \leq n\} \in \mathcal{B}_n$. Therefore $A \cap \{\nu_2 = n\} = A \cap \{\nu_1 \leq n\} \cap \{\nu_2 = n\} \in \mathcal{B}_n$ so $A \in \mathcal{B}_{\nu_2}$. □

LEMMA 3.5.6. Let $\nu_1, \nu_2 \in T$ and $\nu_1 \leq \nu_2$, then

$$\nu = \nu_1 1_{\{\nu_1 = \nu_2\}} + (\nu_1 + 1) 1_{\{\nu_2 > \nu_1\}}$$

is a stopping time such that $\nu_1 \leq \nu \leq \nu_2$ and $|\nu - \nu_1| \leq 1$.

PROOF: $\{\nu = n\} = [\{\nu_1 = n\} \cap \{\nu_2 = n\}] \cap [\{\nu_1 = n-1\} \cap \{\nu_2 \geq n\}] \in \mathcal{B}_n$.

$\square$

Note that if $\nu_1, \nu_2 \in T$ and $\nu_1 \leq \nu_2$ we can use the above construction to find a finite sequence $\{\tau_k\}$ of stopping times such that $\nu_1 = \tau_1 \leq \tau_2 \cdots \leq \tau_n = \nu_2$ and

$$|\tau_{n+1} - \tau_n| \leq 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

LEMMA 3.5.7. If $\tau_1, \tau_2 \in T$ where $\tau_1 \leq \tau_2$ and $|\tau_2 - \tau_1| \leq 1$ then for any $j \in \mathbb{N}$

$$E\left(X_{\tau_2} - X_{\tau_1} | \mathcal{B}_{\tau_1}\right) 1_{\{\tau_1 = j\}} = E(X_{j+1} - X_j | \mathcal{B}_j) 1_{\{\tau_1 = j\} \cap \{\tau_2 = j+1\}}$$

PROOF: By Proposition II-1-3 of [25] we have that

$$E\left(X_{\tau_2} - X_{\tau_1} | \mathcal{B}_{\tau_1}\right) 1_{\{\tau_1 = j\}} = E\left(X_{\tau_2} - X_{\tau_1} | \mathcal{B}_j\right) 1_{\{\tau_1 = j\}}$$

so, since $\{\tau_1 = j\} \cap \{\tau_2 = j+1\} = \{\tau_1 = j\} \cap \{(\tau_2 \leq j)^c\} \in \mathcal{B}_j$,

$$\begin{aligned} E\left(X_{\tau_2} - X_{\tau_1} | \mathcal{B}_{\tau_1}\right) 1_{\{\tau_1 = j\}} &= E\left(X_{\tau_2} - X_{\tau_1} | \mathcal{B}_j\right) 1_{\{\tau_1 = j\}} \\ &= E\left(X_{\tau_2} - X_j 1_{\{\tau_1 = j\}} | \mathcal{B}_j\right) + E\left((X_{j+1} - X_j) 1_{\{\tau_1 = j\} \cap \{\tau_2 = j+1\}} | \mathcal{B}_j\right) \\ &= E(X_{j+1} - X_j | \mathcal{B}_j) 1_{\{\tau_1 = j\} \cap \{\tau_2 = j+1\}}. \end{aligned}$$

$\square$

LEMMA 3.5.8. Let $\tau_1 \leq \tau_2 \leq \cdots$ be a sequence in T such that $|\tau_{n+1} - \tau_n| \leq 1$ for all $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ there exists N such that

$$\sum_{i=1}^{n-1} \left|E\left(X_{\tau_{i+1}} - X_{\tau_i} | \mathcal{B}_{\tau_i}\right)\right| \leq \sum_{j=1}^{N} |E(X_{j+1} - X_j | \mathcal{B}_j)|$$

PROOF: Let $N \geq \tau_n$. We partition Ω into the following sets

$$A_{\xi_1 \dots \xi_n} = \bigcap_{k=1}^{n} \{\tau_k = \xi_k\}\ ,\quad \xi_k \in \{1,\dots,N\} \quad \text{for all} \quad k$$

Consider $\sum_{i=1}^{n-1} \left|E\left(X_{\tau_{i+1}} - X_{\tau_i} \middle| \mathcal{B}_{\tau_i}\right)\right| 1_{A_{\xi_1 \dots \xi_n}}$. By lemma 3.5.7 we can replace $\left|E\left(X_{\tau_{i+1}} - X_{\tau_i} \middle| \mathcal{B}_{\tau_i}\right)\right|$ by $\left|E\left(X_{\xi_{i+1}} - X_{\xi_i} \middle| \mathcal{B}_{\xi_i}\right)\right|$ on $A_{\xi_1 \dots \xi_n}$ for each $i = 1,\dots,n-1$, since $A_{\xi_1 \dots \xi_n} \subseteq [\tau_i = \xi_i]$. So

$$\begin{aligned} \sum_{i=1}^{n-1} \left|E\left(X_{\tau_{i+1}} - X_{\tau_i} \middle| \mathcal{B}_{\tau_i}\right)\right| 1_{A_{\xi_1 \dots \xi_n}} &= \sum_{i=1}^{n-1} \left|E\left(X_{\xi_{i+1}} - X_{\xi_i} \middle| \mathcal{B}_{\xi_i}\right)\right| 1_{A_{\xi_1 \dots \xi_n}} \\ &\leq \sum_{j=1}^{N} \left|E(X_{j+1} - X_j | \mathcal{B}_j)\right| \end{aligned}$$

By taking the union over all possible $A_{\xi_1 \dots \xi_n}$ we get the required result.

□

PROPOSITION 3.5.9. OSAT is valid for sub (super) martingales.

PROOF: Let $\nu_1 \leq \nu_2 \leq \dots$ be a sequence in T. Let $k \in N$ and let $\nu_k = \tau_1 \leq \tau_2 \leq \dots \leq \tau_n = \nu_{k+1}$ be the stopping times as in the note following 3.5.6. Then

$$E\left(X_{\nu_{k+1}} - X_{\nu_k} \middle| \mathcal{B}_{\nu_k}\right) = E\left(\sum_{j=1}^{n-1} E\left(X_{\tau_{j+1}} - X_{\tau_j} \middle| \mathcal{B}_{\tau_j}\right) \middle| \mathcal{B}_{\nu_k}\right).$$

By Lemma 3.5.7, if $\{X_n\}$ were a submartingale, each term $E\left(X_{\tau_{j+1}} - X_{\tau_j} \middle| \mathcal{B}_{\tau_j}\right)$ is positive, so $E\left(X_{\nu_{k+1}} - X_{\nu_k} \middle| \mathcal{B}_{\nu_k}\right) \geq 0$ and $\{X_{\nu_k}\}$ is

a submartingale in $(\Omega, \mathcal{B}, P, \{\mathcal{B}_{\nu_k}\})$. The supermartingale case follows similarly, as does the martingale case.

□

PROPOSITION 3.5.10. OSAT is valid for quasimartingales.

PROOF: Let $\{X_n\}$ be a quasimartingale and $\nu_1 \leq \nu_2 \leq \cdots$ a sequence in T. Let $\tau_1 \leq \tau_2 \leq \cdots$ be a new sequence in T constructed by including all the stopping times between ν_{n+1} and ν_n for each n, as in the note following 3.5.6.

Let $n \in \mathbb{N}$ and $N \in \mathbb{N}$ such that $\tau_{N+1} = \nu_{n+1}$, then

$$\sum_{j=1}^{n} \left|E\left(X_{\nu_{j+1}} - X_{\nu_j} \middle| \mathcal{B}_{\nu_j}\right)\right| \leq \sum_{i=1}^{N} \left|E\left(X_{\tau_{i+1}} - X_{\tau_i} \middle| \mathcal{B}_{\tau_i}\right)\right|$$

Choose N_2 as in Lemma 3.5.8 such that

$$\sum_{i=1}^{N} \left|E\left(X_{\tau_{i+1}} - X_{\tau_i} \middle| \mathcal{B}_{\tau_i}\right)\right| \leq \sum_{k=1}^{N_2} |E(X_{k+1} - X_k | \mathcal{B}_k)|$$

Thus

$$\sum_{j=1}^{n} E|E(X_{\nu_{j+1}} - X_{\nu_j} | \mathcal{B}_{\nu_j})| \leq \sum_{k=1}^{N_2} E|E(X_{k+1} - X_k | \mathcal{B}_k)|$$

$$\leq \sum_{k=1}^{\infty} E|E(X_{k+1} - X_k | \mathcal{B}_k)| < \infty$$

Letting $n \to \infty$ we see that $\{X_{\nu_j}\}$ is a quasimartingale on $(\Omega, \mathcal{B}, P, \{\mathcal{B}_{\nu_j}\})$.

□

PROPOSITION 3.5.11. OSAT is valid for amarts.

PROOF ([13]): We note that if $\{\tau_k\}$ is a sequence of stopping times

on $\{\mathcal{B}_n\}$ and if σ is a stopping time on $\{\mathcal{B}_{\tau_k}\}$, then τ_σ is a stopping time on $\{\mathcal{B}_n\}$ as

$$\{\tau_\sigma = n\} = \bigcup_k (\{\tau_k = n\} \cap \{\sigma = k\}) \in \mathcal{B}_n$$

Now given $\varepsilon > 0$ choose $N \in \mathbb{N}$ so that $|\int (X_\tau - X_{\tau'})| \leq \varepsilon$ if $\tau, \tau' \geq N$. Write $\tau = \lim_k \tau_k$ and τ is a stopping time. Now $X_{\tau_k \wedge N} \to X_{\tau \wedge N}$ as $k \to \infty$ and

$$\int \sup_k |X_{\tau_k \wedge N}| \leq \int \sup_{1 \leq k \leq N} |X_k| < \infty$$

so by the dominated convergence theorem the sequence $\{X_{\tau_k \wedge N}\}$ is an amart. Choose $k \in \mathbb{N}$ so that if σ, σ' are bounded stopping times for $\{\mathcal{B}_{\tau_k}\}$, with $\sigma, \sigma' \geq k$ then

$$|\int X_{\tau_\sigma \wedge N} - \int X_{\tau_{\sigma'} \wedge N}| < \varepsilon$$

Let $\sigma, \sigma' \geq k$, $\tau_\sigma, \tau_{\sigma'}$ are bounded stopping times for $\{\mathcal{B}_n\}$ hence

$$\begin{aligned} |\int X_{\tau_\sigma} - \int X_{\tau_{\sigma'}}| &\leq |\int X_{\tau_\sigma \vee N} - \int X_{\tau_{\sigma'} \vee N}| + |\int X_{\tau_\sigma \wedge N} - X_{\tau_{\sigma'} \wedge N}| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

and thus $\{X_{\tau_k}\}$ is an amart on $(\Omega, \mathcal{B}, P, \{\mathcal{B}_{\tau_k}\})$. □

EXAMPLE 3.5.12. OSTT is not valid for L^1-martingales, MILs or GFTs.

This example, taken from [14], is a process $\{X_n\}$ which is both an L^1-martingale and a MIL, and a stopping time τ such that $\{X_{\tau \wedge n}, \mathcal{B}_{\tau \wedge n}\}$ is not a GFT. Thus OSTT does not hold for any of these processes.

Let $\{A_n\}$ be a sequence of independent events with $P(A_1) = 0$ and $P(A_n) = 1/n^2$ for $n \geq 2$. Define $X_n = n1_{A_n}$, $\mathcal{B}_n = \sigma(X_1,\ldots,X_n)$, for $n \in \mathbb{N}$.

By the Borel-Cantelli Lemma $X_n \to 0$ a.s. and $n > m$ implies $E(X_n|\mathcal{B}_m) - X_m = E(X_n) - X_m \to 0$ a.s. so $\{X_n\}$ is a MIL. Furthermore $\|E(X_n - X_m|\mathcal{B}_m)\|_1 \leq \|X_n\|_1 + \|X_m\|_1 = \frac{1}{n} + \frac{1}{m} \to 0$ as $n > m \to \infty$, and thus $\{X_n\}$ is an L^1-martingale.

Let $\tau(\omega) = \begin{cases} \inf\{n: X_n > 0\} & X_n(\omega) > 0 \text{ for some } n \\ \infty & \text{otherwise.} \end{cases}$

Let $Y_n = X_{n\wedge\tau}$ and $\mathcal{F}_n = \mathcal{B}_{n\wedge\tau}$. Let M be any positive constant and choose $n > m$ so that $\sum_{k=m+1}^{n} \frac{1}{k} \geq M$. Now $A_2^c \cap \ldots \cap A_m^c$ is an atom of $\mathcal{B}_m$, and thus an atom of $\mathcal{F}_m$. Therefore on $A_2^c \cap \ldots \cap A_m^c$, $E(Y_n|\mathcal{F}_m)$ is equal to its average value on the atom. Thus

$$E(Y_n|\mathcal{F}_m) = \frac{1}{m+1} + \sum_{k=m+2}^{n} \frac{1}{k}\left[\prod_{j=m+1}^{k-1}\left(1 - \frac{1}{j^2}\right)\right]$$

$$\geq \frac{1}{2}\sum_{k=m+1}^{n} \frac{1}{k} \geq M/2 \quad \text{on } A_2^c \cap \ldots \cap A_m^c.$$

Since M is arbitrary and Y_m is 0 on $A_2^c \cap \ldots \cap A_m^c$, $\sup_{n\geq m} E(Y_n - Y_m|\mathcal{B}_m) = \infty$ on this set.

The claim is completed by noting that

$$P[\cap A_k^c] = P[\tau = \infty] = \prod_{2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2} > 0$$

so $\{X_{n\wedge\tau}, \mathcal{B}_{n\wedge\tau}\}$ is not a GFT. ∎

EXAMPLE 3.5.13. OSAT is not valid for progressive or eventual

martingales.

Finding a progressive martingale for which OSAT fails is trivial as the "inclusion" criterion is impossible to preserve under sampling. Let $\{X_n\}$ be a progressive martingale with $P[X_1 = 0] = 0$ and $P[E(X_2 - X_1 | \mathcal{B}_1) = 0] < 1$. Now add to the beginning of the sequence $X_{-2} = X_0 = 0$ and $X_{-1} = 1$ and $\mathcal{B}_{-2} = \mathcal{B}_{-1} = \mathcal{B}_0 = \{\phi, \Omega\}$.

Now $\{X_{-2}, X_{-1}, X_0, X_1, \ldots\}$ is a progressive martingale, but if we let $\tau_1 = -2$, $\tau_2 = 0$ and $\tau_k = k-2$ for $k \geq 3$ then we have

$$\left\{E\left(X_{\tau_2} - X_{\tau_1} | \mathcal{B}_{\tau_1}\right) = 0\right\} = \Omega \not\subseteq \left\{E\left(X_{\tau_4} - X_{\tau_3} | \mathcal{B}_{\tau_3}\right) = 0\right\}$$
$$= \{E(X_2 - X_1 | \mathcal{B}_1) = 0\}$$

so that $\left\{X_{\tau_k}, \mathcal{B}_{\tau_k}\right\}$ is not a progressive martingale.

In the eventual martingale case, consider $\{X_n\}$ as constructed in 3.3.13. It is clear that we can find $\tau_1 \leq \tau_2 \leq \ldots$ such that $X_{\tau_{2n}} = 1$ and $X_{\tau_{2n-1}} = -1$, $n \in \mathbb{N}$ and thus $\{X_{\tau_k}\}$ is not an eventual martingale.

□

PROPOSITION 3.5.14. OSTT is valid for a progressive and eventual martingales.

PROOF: We begin with a simple identity for any process $\{X_n\}$ and a stopping time τ.

$$E(X_{n\wedge\tau} - X_{(n-1)\wedge\tau} | \mathcal{B}_{(n-1)\wedge\tau}) = \sum_{i=1}^{\infty} E(X_{n\wedge\tau} - X_{(n-1)\wedge\tau} | \mathcal{B}_i) 1_{\{(n-1)\wedge\tau = i\}}$$

$$= \sum_{i=1}^{n-1} E(X_{n\wedge\tau} - X_{(n-1)\wedge\tau} | \mathcal{B}_i) 1_{\{\tau = i\}} + E(X_{n\wedge\tau} - X_{(n-1)\wedge\tau} | \mathcal{B}_{n-1}) 1_{\{\tau \geq n\}}$$

$$= E((X_n - X_{n-1}) 1_{\{\tau \geq n\}} | \mathcal{B}_{n-1}) = E(X_n - X_{n-1} | \mathcal{B}_{n-1}) 1_{\{\tau \geq n\}}.$$

Thus $\{E(X_{n\wedge\tau}-X_{(n-1)\wedge\tau}|\mathcal{B}_{(n-1)\wedge\tau})=0\} = \{\tau<n\} \cup \{E(X_n-X_{n-1}|\mathcal{B}_{n-1})=0\}$.

Now if $\{X_n\}$ were a progressive martingale

$$\{E(X_{n\wedge\tau}-X_{(n-1)\wedge\tau}|\mathcal{B}_{(n-1)\wedge\tau})=0\} = \{\tau<n\} \cup \{E(X_n-X_{n-1}|\mathcal{B}_{n-1})=0\}$$
$$\subseteq \{\tau<n+1\} \cup \{E(X_{n+1}-X_n|\mathcal{B}_n)=0\} = \{E(X_{(n+1)\wedge\tau}-X_{n\wedge\tau}|\mathcal{B}_{n\wedge\tau})=0\}$$

Also $\{E(X_n-X_{n-1}|\mathcal{B}_{n-1})=0\} \subseteq \{E(X_{n\wedge\tau}-X_{(n-1)\wedge\tau}|\mathcal{B}_{(n-1)\wedge\tau})=0\}$ implies that

$$P[\{E(X_{n\wedge\tau}-X_{(n-1)\wedge\tau}|\mathcal{B}_{(n-1)\wedge\tau})=0\}] \uparrow 1$$

Thus $\{X_{n\wedge\tau},\mathcal{B}_{n\wedge\tau}\}$ is a progressive martingale.

Similarly if $\{X_n\}$ were an eventual martingale then

$$\liminf \{E(X_n-X_{n-1}|\mathcal{B}_{n-1})=0\} \subseteq \liminf \{E(X_{n\wedge\tau}-X_{(n-1)\wedge\tau}|\mathcal{B}_{(n-1)\wedge\tau})=0\}$$
$$\Rightarrow P[\liminf \{E(X_{n\wedge\tau}-X_{(n-1)\wedge\tau}|\mathcal{B}_{(n-1)\wedge\tau})=0\}] = 1$$

so $\{X_{n\wedge\tau},\mathcal{B}_{n\wedge\tau}\}$ is an eventual martingale. $\square$

Transforms

PROPOSITION 2.5.15. TT is valid for martingales and eventual martingales.

PROOF: Let $A_n = \{E((U*X)_{n+1} - (U*X)_n|\mathcal{B}_n)=0\}$. The identity

$$E((U*X)_{n+1} - (U*X)_n|\mathcal{B}_n) = U_{n+1}E(X_{n+1}-X_n|\mathcal{B}_n)$$

tells us that $\{E(X_{n+1}-X_n|\mathcal{B}_n)=0\} \subseteq A_n$, $n \in \mathbb{N}$. If $\{X_n\}$ were a martingale, then $P[E(X_{n+1}-X_n|\mathcal{B}_n)=0] = 1$ so $P(A_n) = 1$ for all $n \in \mathbb{N}$

and thus $\{(U*X)_n\}$ is a martingale.

Similarly if $\{X_n\}$ were an eventual martingale

$$1 = P[\liminf \{E(X_{n+1}-X_n|\mathcal{B}_n) = 0\}] \leq P[\liminf A_n] = 1$$

tells us that $\{(U*X)_n\}$ is an eventual martingale.

□

EXAMPLE 3.5.16. TT is not valid for sub or supermartingales.

It is clear that if $\{X_n\}$ is a submartingale which is not a martingale then letting $U_n = -1$ for all $n \in \mathbb{N}$, gives us a transformed process which is not a submartingale. Similarly for the supermartingale case.

□

EXAMPLE 3.5.17. TT is not valid for progressive martingales.

If $\{X_n\}$, $n \in \mathbb{N}_+$ is a progressive martingale but not a martingale then if $U_1 = 0$ and $U_2 = 1$ the transformed process is not a progressive martingale as

$$\Omega = A_1 \supseteq A_2 = \{E(X_2-X_1|\mathcal{B}_1) = 0\}$$

where A_n is defined as in 3.5.15. Since the containment is proper, $\{(U*X)_n\}$ is not a progressive martingale.

□

EXAMPLE 3.5.18. TT is not valid for quasimartingales, amarts, L^1-martingales, MILs or GFTs.

We will show that the transform theorem fails for all of these processes by giving a quasimartingale $\{X_n\}$ and a sequence $\{U_n\}$ such that the transformed process is not even a GFT. Let $(\Omega,\mathcal{B},P)$ be any

probability space and let $X_n = 1/2^n$, $\mathcal{B}_n = \{\phi, \Omega\}$ for each $n \in \mathbb{N}_+$. Then $\{X_n\}$ is a quasimartingale as

$$\sum E(|E(X_{n+1} - X_n | \mathcal{B}_n)|) = \sum \frac{1}{2^n} < \infty$$

If we let $U_0 = 0$ and $U_n = -2^n$ for each $n \in \mathbb{N}$, we have

$$(U*X)_n = \sum_{k=1}^{n} U_k/2^k + U_0 = n$$

so $\{(U*X)_n\}$ is clearly not a GFT.

□

CHAPTER 4

REAL-VALUED PROCESSES INDEXED BY $\mathbb{N}_+^2$

4.1 Background

Indexing martingales by a directed set is not a new idea, as it goes back to the beginnings of martingale theory. However, in going from a totally ordered index set to a partially ordered one, we cannot preserve the convergence properties of these processes. For example, we've seen that an L^1-bounded martingale $\{X_n, n \in \mathbb{N}\}$ must converge a.s., but there are uniformly integrable martingales $\{X_n, n \in \mathbb{N}_+^2\}$ which do not [6].

Work has been done on finding conditions under which a martingale $\{X_n, n \in \mathbb{N}_+^2\}$ will converge almost surely. Cairoli, in [6] showed that an $L \log L$ bounded martingale converges a.s., and in [7] gave other conditions under which a martingale converges a.s. Krickeberg [18] and Gabriel [17] each gave a proof that every L^1-bounded martingale indexed by a directed set converges a.s. provided that the σ-algebras $\{\mathcal{B}_n\}$ satisfy the Vitali condition. Walsh [33] showed that an L^1-bounded strong martingale indexed by $\mathbb{N} \times \mathbb{N}$ or $\mathbb{R} \times \mathbb{R}$, converges a.s. and Millet [19] extended this to strong submartingales. Chatterji [10] also considered the a.s. convergence of martingales indexed by a directed set.

In this chapter we extend the martingale-like processes to processes indexed by $\mathbb{N}_+^2$ and consider their properties.

4.2 Relationships Between the Processes

Put the following order on $\mathbb{N}_+^2$. If $s = (s_1,s_2)$ and $t = (t_1,t_2)$,

then $s \leq t$ if $s_1 \leq t_1$ and $s_2 \leq t_2$. For $s \in \mathbb{N}_+^2$ define $H_s = \{n \in \mathbb{N}_+^2 | n \leq s\}$ and if $(1,1) \leq s$ define $s - 1 = (s_1-1, s_2-1)$.

Let $(\Omega, \mathcal{B}, P, \{\mathcal{B}_n, n \in \mathbb{N}_+^2\})$ be a filtered probability space such that $s \leq t \Rightarrow \mathcal{B}_s \subseteq \mathcal{B}_t$. A net of random variables $\{X_n, n \in \mathbb{N}_+^2\}$ is called *adapted* if X_n is $\mathcal{B}_n$-measurable for each $n \in \mathbb{N}_+^2$.

If $n = (n_1, n_2) \in \mathbb{N}_+^2$ we define

$$d_n = X_{(n_1+1, n_2+1)} - X_{(n_1, n_2+1)} - X_{(n_1+1, n_2)} + X_{(n_1, n_2)}$$

and note that if $(1,1) \leq s$, then

$$\sum_{n \in H_{s-1}} d_n = X_s - X_{(s_1, 0)} - X_{(0, s_2)} + X_{(0,0)}$$

DEFINITION 4.2.1. A function $\tau = (\tau_1, \tau_2): \Omega \to \mathbb{N}_+^2 \cup \{\infty\}$ is called a *stopping time* if

$$\{\omega | \tau(\omega) = n\} \in \mathcal{B}_n \quad \text{for each} \quad n \in \mathbb{N}_+^2$$

and is called an *axis stopping time* if

$$\{\omega | \tau(\omega) = n\} \in \mathcal{B}_{(n_1, 0)} \cap \mathcal{B}_{(0, n_2)} \quad \text{for each} \quad (n_1, n_2) \in \mathbb{N}_+^2$$

We also define $\tau_1 = (\tau_1, 0)$ and $\tau_2 = (0, \tau_2)$. We let T [resp. AT] denote the net of all bounded [axis] stopping times with the natural ordering. □

In each of the following definitions, the process $\{X_n, n \in \mathbb{N}_+^2\}$ is assumed to be adapted and in L^1.

DEFINITION 4.2.2. A process $\{X_n\}$ is called a *sub* [resp. *super*] *martingale* if $E(X_t - X_s | \mathcal{B}_s) \geq$ [resp. $\leq$] 0 if $s \leq t \in \mathbb{N}_+^2$. In case of equality $\{X_n\}$ is called a martingale.

DEFINITION 4.2.3. A process $\{X_n\}$ is called a *sequential quasi-martingale* [abbreviated S. Quasi.] if whenever $\{t_n\}$ is a non-decreasing cofinal sequence in $\mathbb{N}_+^2$, then $\{X_{t_n}, \mathcal{B}_{t_n}\}$ is a one parameter quasi-martingale.

DEFINITION 4.2.4. A process $\{X_n\}$ is called a *sequential amart* [abbreviated S. Amart] if whenever $\{t_n\}$ is as above, $\{X_{t_n}, \mathcal{B}_{t_n}\}$ is a one parameter amart.

DEFINITION 4.2.5. A process $\{X_n\}$ is called a *sequential martingale in the limit* [abbreviated S.MIL] if whenever $\{t_n\}$ is as above, $\{X_{t_n}, \mathcal{B}_{t_n}\}$ is a one parameter MIL.

DEFINITION 4.2.6. A process $\{X_n\}$ is called a *quasimartingale* if $\{X_{(0,n)}, \mathcal{B}_{(0,n)}\}$ and $\{X_{(n,0)}, \mathcal{B}_{(n,0)}\}$ are one parameter quasimartingales and

$$\sum_n E|E(d_n|\mathcal{B}_n)| < \infty$$

DEFINITION 4.2.7. A process $\{X_n\}$ is called an *amart* if the net $\{E(X_\tau), \tau \in T\}$ converges, and an *axis amart* if the net $\{E(X_\tau), \tau \in AT\}$ converges.

DEFINITION 4.2.8. A process $\{X_n\}$ is called a *martingale in the limit* [abbreviated MIL] if $\{Y_m, m \in \mathbb{N}_+^2\}$ converges to 0 a.s. where

$$Y_m = \sup_{n \geq m} |E(X_n - X_m | \mathcal{B}_m)|$$

DEFINITION 4.2.9. A process $\{X_n\}$ is called a *game which gets fairer with time* [abbreviated GFT] if for each $\varepsilon > 0$ the net $\{y_m, m \in \mathbb{N}_+^2\}$ converges to 0 where

$$y_m = \sup_{n \geq m} P[|E(X_n - X_m | \mathcal{B}_m)| > \varepsilon]$$

The following lemmas will be useful.

LEMMA 4.2.10. If $\{x_n, n \in \mathbb{N}_+^2\}$ is a net in a complete metric space, then $\{x_n\}$ converges if and only if, whenever $\{n_k\}$ is a non-decreasing cofinal subsequence of $\mathbb{N}_+^2$, $\{x_{n_k}\}$ converges.

PROOF: ($\Rightarrow$) Let $\varepsilon > 0$ and suppose x_n converges to x. Then there exists $N \in \mathbb{N}_+^2$ such that $n \geq N$ implies $d(x_n, x) \leq \varepsilon$. Let $\{n_k\}$ be an increasing cofinal subsequence of $\mathbb{N}_+^2$. Then there exists k_1 such that $n_{k_1} > N$, so $k \geq k_1$ implies $d(x_{n_k}, x) \leq \varepsilon$. Thus $\{x_{n_k}\}$ converges to x.

($\Leftarrow$) Suppose that $\{x_{n_k}\}$ converges for all $\{n_k\}$ increasing cofinal sequences in $\mathbb{N}_+^2$, and suppose that $\{x_n\}$ does not converge. Then $\{x_n\}$ is not Cauchy so we can find $\varepsilon > 0$ such that for any $n \in \mathbb{N}_+^2$, there exist $n_1, n_2 \geq n$ such that $d(x_{n_1}, x_{n_2}) > \varepsilon$. Since $d(x_n, x_{n_1}) + d(x_n, x_{n_2}) \geq d(x_{n_1}, x_{n_2}) > \varepsilon$, one of $d(x_n, x_{n_1})$ or $d(x_n, x_{n_2})$ is greater than $\varepsilon/2$.

Let $\{s_k\}$ be an enumeration of $\mathbb{N}_+^2$. Define an increasing cofinal sequence $\{r_k\}$ by letting $r_1 = s_1$, and then inductively by

$$r_{2k} > r_{2k-1} \quad \text{such that} \quad d(x_{r_{2k}}, x_{r_{2k-1}}) > \varepsilon/2$$

$$r_{2k+1} > r_{2k} \qquad \text{and} \qquad r_{2k+1} > s_k$$

for all $k \in \mathbb{N}$. Then $\{x_{r_k}\}$ is not Cauchy and cannot converge which is a contradiction.

□

LEMMA 4.2.11. $\{X_n, n \in \mathbb{N}_+^2\}$ is a GFT if and only if $\{X_{n_k}, k \in N\}$ is a one parameter GFT whenever $\{n_k\}$ is an increasing cofinal subsequence in $\mathbb{N}_+^2$.

PROOF: $(\Longrightarrow)$ Let $\{m_k\}$ be an increasing cofinal subsequence in $\mathbb{N}_+^2$. Then for $\varepsilon > 0$ and $k \in \mathbb{N}$

$$\sup\left\{P\left[\left|E\left(X_n - X_{m_k} \middle| \mathcal{B}_{m_k}\right)\right| > \varepsilon\right] \middle| n \in \mathbb{N}_+^2,\ n \geq m_k\right\}$$
$$\geq \sup\left\{P\left[\left|E\left(X_n - X_{m_k} \middle| \mathcal{B}_{m_k}\right)\right| > \varepsilon\right] \middle| n \in \{m_k\}, n \geq m_k\right\} \geq 0$$

By the definition of GFT and 4.2.10 the sequence on the left hand side converges to zero as $k \to \infty$, and thus so must the right hand side. Therefore $\{X_{m_k}\}$ is a GFT.

$(\Longleftarrow)$ If $\{X_n, n \in \mathbb{N}_+^2\}$ is not GFT then by definition and 4.2.10 we can find $\varepsilon > 0$ and an increasing cofinal sequence $\{m_k\} \in \mathbb{N}_+^2$ such that

$$y_{m_k} = \sup_{n \geq m_k} P\left[\left|E\left(X_n - X_{m_k} \middle| \mathcal{B}_{m_k}\right)\right| > \varepsilon\right]$$

does not converge to zero. Therefore there exists $\delta > 0$ such that

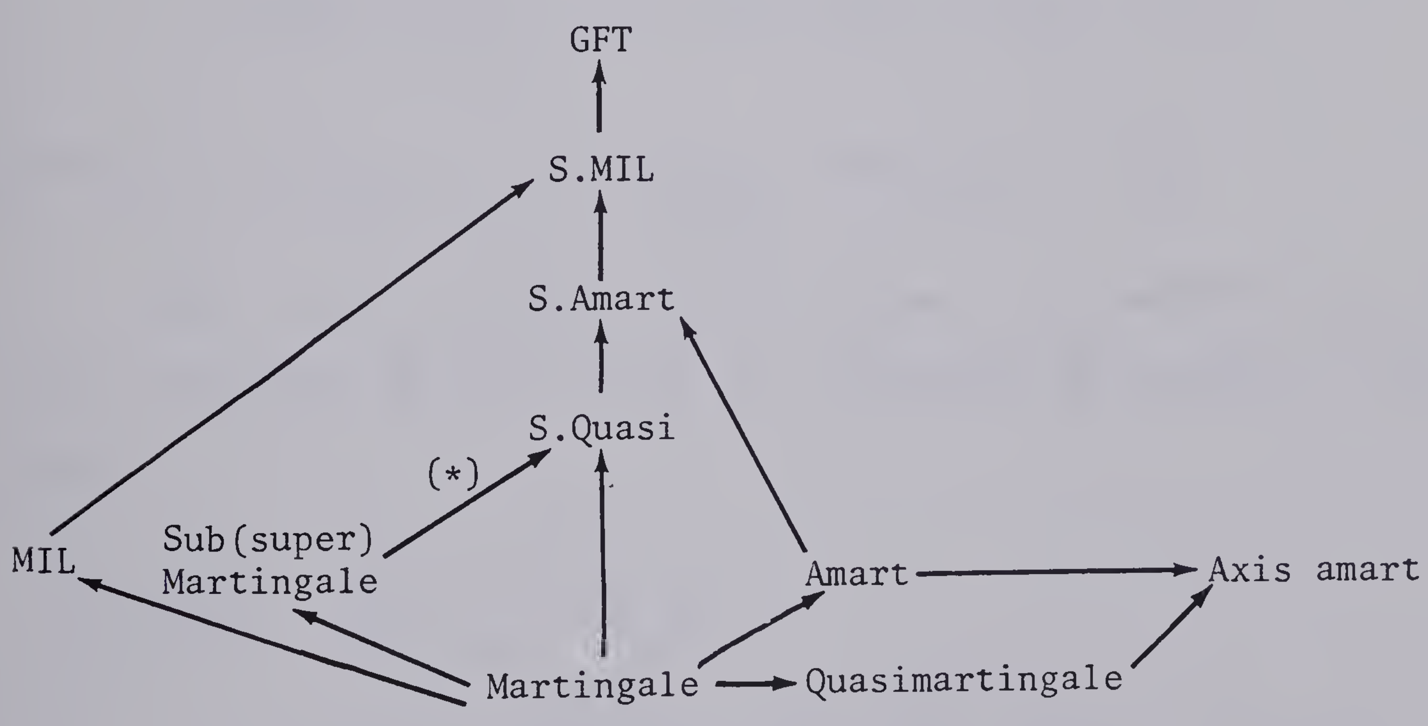
for each $k \in \mathbb{N}$, there exists $k' \geq k$ so that

$$P\left[\left|E\left(X_{n_{k'}} - X_{m_{k'}} \middle| \mathcal{B}_{m_{k'}}\right)\right| > \varepsilon\right] > \delta$$

for some $m_{k'} \in \{m_k\}$ and $n_{k'} \in \mathbb{N}_+^2$. Since $\{m_k\}$ is cofinal it is clear that we can extract a new cofinal non-decreasing sequence $\{\ell_k\}$ such that $\{X_{\ell_k}\}$ is not a GFT. This contradicts the hypothesis, so $\{X_n, n \in \mathbb{N}_+^2\}$ is a GFT. □

The following diagram illustrates some of the relationships between the processes defined above. A number of proofs are given afterwards.

DIAGRAM 4.2.12



(*) if and only if $\{X_n\}$ is L^1-bounded

REMARK: The vertical arrows and the ones involving the submartingales and supermartingales are trivial to verify. Simply consider increasing cofinal sequences and the corresponding one parameter relationships.

Amart $\to$ Axis amart, Amart $\to$ S.Amart and MIL $\to$ S.MIL are trivial.

Martingale $\longrightarrow$ Quasimartingale is trivial as $E(d_n|\mathcal{B}_n) = 0$ for each $n \in \mathbb{N}_+^2$.

□

PROPOSITION 4.2.13. Every quasimartingale is an axis amart.

PROOF: Let $\{X_n, \mathcal{B}_n\}$ $n \in \mathbb{N}_+^2$ be a quasimartingale. By definition the axis processes are quasimartingales and thus are one parameter amarts (3.2.11). So if $\varepsilon > 0$, we can choose $n_1, n_2 \in \mathbb{N}$ so that if ξ_1, $\xi_2 \geq n_1$ are bounded stopping times for $\{X_{(n,0)}, \mathcal{B}_{(n,0)}\}$ $n \in \mathbb{N}_+$, and $\psi_1, \psi_2 \geq n_2$ are bounded stopping times for $\{X_{(0,n)}, \mathcal{B}_{(0,n)}\}$ $n \in \mathbb{N}_+$, then

$$|\int \left(X_{\xi_1} - X_{\xi_2}\right)| < \varepsilon/3 \quad \text{and} \quad |\int \left(X_{\psi_1} - X_{\psi_2}\right)| < \varepsilon/3$$

Now let $m \geq (n_1, n_2)$ so that $\sum_{n \in H_{m-1}^c} E|E(d_n|\mathcal{B}_n)| < \varepsilon/3$.

Let $\tau \in AT$ such that $m \leq \tau$. Since τ is bounded we can find $\tau \leq M = (M_1, M_2)$. We let $k = (k_1, k_2)$ stand for any element of $\mathbb{N}_+^2$. Consider

$$\begin{aligned}
|\int (X_M - X_\tau)| &= |\sum_k \int_{\{\tau=k\}} (X_M - X_k)| \\
&= |\sum_k \int_{\{\tau=k\}} \Bigg(\Big(\sum_{j \in (H_{M-1} \cap H_{k-1}^c)} d_j\Big) - X_{(M_1,0)} + X_{(k_1,0)} \\
&\qquad - X_{(0,M_2)} + X_{(0,k_2)}\Bigg)| \\
&\leq |\sum_k \int_{\{\tau=k\}} \Big(\sum_{j \in H_{M-1} \cap H_{k-1}^c} d_j\Big)| + |\sum_{k_1} \int_{\{\tau_1=(k_1,0)\}} \Big(X_{(M_1,0)} - X_{(k_1,0)}\Big)| \\
&\qquad + |\sum_{k_2} \int_{\{\tau_2=(0,k_2)\}} \Big(X_{(0,M_2)} - X_{(0,k_2)}\Big)|
\end{aligned}$$

For $j \in H_{k-1}^c$, $j \not\leq k-1$ so $j_1 > k_1 - 1$ or $j_2 > k_2 - 1$ and thus $j_1 \geq k_1$ or $j_2 \geq k_2$. Thus $\{\tau = k\} \in \mathcal{B}_{(k_1,0)} \cap \mathcal{B}_{(0,k_2)} \subseteq \mathcal{B}_j$. So

$$|E(X_M - X_\tau)|$$

$$\leq \left| \sum_k \int_{\{\tau=k\}} \left(\sum_{j \in H_{M-1} \cap H_{k-1}^c} E(d_j | \mathcal{B}_j) \right) \right| + \left| E\left(X_{(M_1,0)} - X_{\tau_1} \right) \right| + \left| E\left(X_{(0,M_2)} - X_{\tau_2} \right) \right|$$

$$\leq \sum_k \int_{\{\tau=k\}} \left(\sum_{j \in H_{M-1} \cap H_{k-1}^c} |E(d_j | \mathcal{B}_j)| \right) + 2\varepsilon/3$$

and since $\tau \geq m$

$$\leq \sum_k \int_{\{\tau=k\}} \left(\sum_{j \in H_{M-1} \cap H_{m-1}^c} |E(d_j | \mathcal{B}_j)| \right) + 2\varepsilon/3$$

$$= E\left(\sum_{j \in H_{M-1} \cap H_{m-1}^c} |E(d_j | \mathcal{B}_j)| \right) + 2\varepsilon/3$$

$$= \sum_{j \in H_{M-1} \cap H_{m-1}^c} E|E(d_j | \mathcal{B}_j)| + 2\varepsilon/3$$

$$\leq \sum_{j \in H_{m-1}^c} E|E(d_j | \mathcal{B}_j)| + 2\varepsilon/3 \leq \varepsilon/3 + 2\varepsilon/3 = \varepsilon.$$

Thus $\{E(X_\tau), \tau \in AT\}$ converges, so $\{X_n\}$ is an axis amart. □

EXAMPLE 4.2.14. We will construct a quasimartingale which is not a GFT. We begin by defining a one parameter martingale. Let Ω be the Lebesque space (3.2.18) with the filtration $\mathcal{F}_n = \sigma\{[j/2^n, j+1/2^n) \mid j = 0,1,\ldots,2^n-1\}$ $n \in \mathbb{N}_+$. We define $\{Y_n\}$ inductively, letting $Y_0 = 0$, $Y_1 = -1_{[0,\frac{1}{2})} + 1_{[\frac{1}{2},1)}$ and if $n \geq 1$, and B is an atom of $\mathcal{F}_n$, if $Y_n = k$ on B let Y_{n+1} take the values $4k$ and $-2k$ on the "halves" of B in $\mathcal{F}_{n+1}$. Therefore

$$\int_B Y_{n+1} = \frac{1}{2}\int_B (4k-2k) = \int_B k = \int_B Y_n$$

so that $\{Y_n, F_n\}$ $n \in \mathbb{N}_+$ is a martingale. Also for $n \in \mathbb{N}$, $|Y_n| \geq 2^{n-1}$ pointwise. Define a sequence of constants $a_0 = 1$ and $a_n = (2^n E(|Y_{n-1}|))^{-1}$ for $n \in \mathbb{N}$.

Consider the process $\{X_n, \mathcal{B}_n\}$ $n \in \mathbb{N}_+^2$, where $\mathcal{B}_{(n_1,n_2)} = F_{n_1}$ and $X_{(n_1,n_2)} = 0$ if $n_1 < n_2$, $X_{(n_1,n_2)} = a_{n_2} Y_{n_1}$ otherwise. Note that $\{X_{(0,n)}\}$ and $\{X_{(n,0)}\}$ are one parameter martingales. Now fix n_1 and consider $n \in \{(n_1,n_2) | n_2 \in \mathbb{N}_+\}$

(i) if $n_2 < n_1$ then

$$E(d_n|\mathcal{B}_n) = E\left(X_{(n_1+1,n_2+1)} - X_{(n_1+1,n_2)} - X_{(n_1,n_2+1)} + X_{(n_1,n_2)} | \mathcal{B}_{(n_1,n_2)}\right)$$

$$= \left(a_{n_2+1} - a_{n_2}\right) E\left(Y_{n_1+1} - Y_{n_1} | F_{n_1}\right) = 0$$

(ii) if $n_2 = n_1$ then $E(d_n|\mathcal{B}_n)$

$$= -a_{n_2} E\left(Y_{n_1+1} - Y_{n_1} | F_{n_1}\right) + a_{n_2+1} E\left(Y_{n_1+1} | F_{n_1}\right)$$

$$= a_{n_2+1} Y_{n_1} = a_{n_1+1} Y_{n_1}$$

(iii) if $n_2 = n_1 + 1$ then $E(d_n|\mathcal{B}_n)$

$$= -a_{n_2} E\left(Y_{n_1+1} | F_{n_1}\right) = -a_{n_1+1} Y_{n_1}$$

(iv) if $n_2 > n_1 + 1$ then $E(d_n|\mathcal{B}_n) = 0$

Thus

$$\sum_n E|E(d_n|\mathcal{B}_n)| = \sum_{n_1} 2 \cdot E|a_{n_1+1}Y_{n_1}|$$

$$= \sum_{n_1} 2a_{n_1+1}E|Y_{n_1}|$$

$$= \sum_{n_1} 1/2^{n_1} < \infty$$

so that $\{X_n\}$ is a quasimartingale.

Now if $(n_1,n_2) \in \mathbb{N}_+^2$, choose $t > n_1$ so $|a_{n_2+1} - a_{n_2}|2^{t-1} > 1$. Then

$$\left|E\left(X_{(t,n_2+1)} - X_{(t,n_2)}\middle|\mathcal{B}_{(t,n_2)}\right)\right| = |a_{n_2+1} - a_{n_2}||E(Y_t|F_t)|$$

$$= |a_{n_2+1} - a_{n_2}| \cdot |Y_t|$$

$$\geq |a_{n_2+1} - a_{n_2}| \cdot 2^{t-1} > 1$$

pointwise, so $\{X_n\}$ cannot be a GFT.

□

EXAMPLE 4.2.15. Here we will construct a sequential quasimartingale which is neither an axis amart nor a MIL.

Let Ω be the Lebesque space and $F_n = \sigma\{[j/n, j+1/n)|j=0,\ldots,n-1\}$ for each $n \in \mathbb{N}$.

For $k \geq 1$ let $S_k = \left\{(n_1,n_2) \in \mathbb{N}_+^2 \;\middle|\; \begin{array}{l} 2^{k-1} \leq n_1 + 1 < 2^k \\ 2^{k-1} \leq n_2 + 1 < 2^k \end{array} \text{ and} \right\}$

These are squares along the diagonal, the length of a side of S_k being 2^{k-1}. If $n = (n_1,n_2) \in \mathbb{N}_+^2$, let $\mathcal{B}_n = F_{2^{2k-2}}$, where

$$k = \max\{k_1,k_2\} \quad \text{where} \quad \begin{array}{l} 2^{k_1-1} \le n_1 + 1 < 2^{k_1} \\ 2^{k_2-1} \le n_2 + 1 < 2^{k_2} \end{array}$$

Notice that there are $(2^{k-1})^2 = 2^{2k-2}$ points in S_k. Let ϕ_k denote a one-to-one map from S_k to $\{0,1,2,\ldots,2^{2k-2}-1\}$.

If $n \in S_k$ let $X_n = 1_{[j/2^{2k-2},\ j+1/2^{2k-2})}$ where $j = \phi_k(n)$ for all $k \ge 1$.

If $n \notin \cup S_k$ let $X_n = 0$.

We will now show that $\{X_n, \mathcal{B}_n\}$ $n \in \mathbb{N}^2$ is a sequential quasi-martingale. Let $\{n_i\}$ be an increasing cofinal sequence in $\mathbb{N}_+^2$ and note that it can only intersect S_k at most at $2 \cdot 2^{k-1}$ places (2^{k-1} vertically and 2^{k-1} horizontally).

Therefore

$$\sum_i E|X_{n_i}| = \sum_{k=1}^{\infty} \sum_{n_i \in S_k} E|X_{n_i}| \le \sum_{k=1}^{\infty} 2 \cdot 2^{k-1}/(2^{k-1})^2$$

$$= \sum_{k=1}^{\infty} 2/2^{k-1} < \infty$$

Thus $$\sum_i E\left|E\left(X_{n_{i+1}} - X_{n_i} \middle| \mathcal{B}_{n_i}\right)\right| \le \sum_i E|X_{n_{i+1}}| + \sum_i E|X_{n_i}|$$

$$\le 4 \cdot \sum_k 1/2^{k-1} < \infty$$

so $\{X_n\}$ is a sequential quasimartingale.

We now show that $\{X_n\}$ is not an axis amart. Let $m = (m_1,m_2) \in \mathbb{N}_+^2$ and find i so that $m < n$ for each $n \in S_i$. Define a stopping time τ_1 as follows, For each $n \in S_i$

$$\tau_1 = n \quad \text{on} \quad [j/2^{2i-2},\ j+1/2^{2i-2}) \quad \text{where} \quad j = \phi_i(n)$$

Note that $\{\tau_1 = n\} \in \mathcal{B}_n = \mathcal{B}_{(n_1,0)} \cap \mathcal{B}_{(0,n_2)}$ so $\tau_1 \in AT$. Also

$$\int X_{\tau_1} = \sum_{j=0}^{2^{2i-2}} \int 1_{[j/2^{2i-2},\ j+1/2^{2i-2})} = \int 1 = 1$$

Now let $\tau_2 = (m_1, 2^i)$. Again $\tau_2 \in AT$ and since $(m_1, 2^i) \notin \cup S_k$ we have $\int X_{\tau_2} = 0$. Thus we have $m \leq \tau_1$ and $m \leq \tau_2$ where τ_1, $\tau_2 \in AT$ and $\int (X_{\tau_1} - X_{\tau_2}) = 1$. Thus $\{E(X_\tau), \tau \in AT\}$ does not converge so $\{X_n\}$ is not an axis amart.

To show that $\{X_n\}$ is not a MIL, let $\omega \in \Omega$ and $m = (m_1, m_2) \in \mathbb{N}_+^2$. Let i be as above and let $n = (n_1, n_2) \in S_i$ so that

$$\omega \in [j/2^{2i-2},\ j+1/2^{2i-2}) \quad j = \phi_i(n)$$

Now let $t = (2^i, n_2)$. Then $m \leq n \leq t$ and

$$|E(X_t - X_n | \mathcal{B}_n)|(\omega) = |X_n|(\omega) = 1$$

so that the net $Y_m = \sup_{n \geq m} |E(X_n - X_m | \mathcal{B}_m)|$ converges to zero nowhere on Ω. Thus $\{X_n\}$ is not a MIL.

□

4.3 <u>Convergence</u>

PROPOSITION 4.3.1. Every uniformly integrable GFT converges in L^1.

PROOF: Let $\{n_k\}$ be an increasing cofinal sequence in N_+^2. By 4.2.11 $\{X_{n_k}\}$ is a GFT and since it is uniformly integrable it

converges in L^1. Since L^1 is a complete metric space, by applying 4.2.10 we see that $\{X_n\}$ converges in L^1.

$\square$

EXAMPLE 4.3.2. Let $Y \in L^1$ with $Y \neq 0$ a.s. but $E(Y) = 0$. Let $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\{\mathcal{F}_n\}$ $n \in \mathbb{N}$ be an increasing sequence of σ-algebras such that $\sigma(Y) \subseteq \sigma(\cup \mathcal{F}_n)$. Then $Y_n = E(Y|\mathcal{F}_n)$ is a one parameter martingale which converges to Y in L^1.

Let $X_{(n_1,n_2)} = (-1)^{n_2} Y_{n_1}$, $\mathcal{B}_{(n_1,n_2)} = \mathcal{F}_{n_1}$ for $n \in \mathbb{N}_+^2$.

Then $\{X_{(0,n)}\}$ and $\{X_{(n,0)}\}$ are one-parameter quasimartingales and

$$E(d_n|\mathcal{B}_n) = \left((-1)^{n_2+1} E\left(Y_{n_1+1} - Y_{n_1} | \mathcal{F}_{n_1}\right)\right) - \left((-1)^{n_2} E\left(Y_{n_1+1} - Y_{n_1} | \mathcal{F}_{n_1}\right)\right)$$

$$= 0 \quad \text{for all} \quad n \in \mathbb{N}_+^2$$

so that $\{X_n\}$ is a quasimartingale. Also, by 2.3.5 $\{X_n\}$ is uniformly integrable but $\{X_n\}$ clearly does not converge in L^1.

$\square$

The interested reader may note (using Diagram 4.2.12) that 4.3.1 and 4.3.2 give a complete picture of the relation between L^1 convergence and the hypothesis of uniform integrability.

CHAPTER 5

BANACH-VALUED PROCESSES INDEXED BY $\mathbb{N}$

5.1 Background

Generalizing martingale results to vector-valued random variables began as soon as a theory of integration and conditional expectation was developed for such functions. The groundwork in the area of Banach-valued martingales was laid in the 1960's by Scalora [29] and Chatterji [9]. Chatterji proved the remarkable result that the following are equivalent for a Banach space E

(i) Every L_E^1-bounded E-valued martingale converges to a random variable $X \in L_E^1$ almost surely

(ii) Every uniformly integrable E-valued martingale converges in L_E^1

(iii) the vector-valued Radon-Nikodym theorem holds in E.

He also gave examples of spaces where (i)-(iii) hold, for instance if E is reflexive, or if E is a separable dual of a Banach space, or if E is a weakly complete space with a separable dual.

The only martingale-like processes whose vector-valued counterparts have been explored in any detail are the amart and the quasimartingale.

In the early 1970's, Pop-Stojanovič [27], [28] proved some decomposition results for vector-valued quasimartingales.

The vector-valued amart was introduced in 1975 in a paper by Chacon and Sucheston [8], in which it was proven that if E has the

Radon-Nikodym property and E^* is separable, and if $\{X_n\}$ is an E-valued amart of class B, that is if $\sup E(\|X_\tau\|) < \infty$, then $\{X_n\}$ converges weakly a.s.

A year later Bellow [3] showed that this conclusion could not be strengthened to strong convergence by proving that E is finite dimensional if and only if every E-valued amart of class B converges strongly pointwise. Edgar and Sucheston [15] then showed that the condition of class B could not be weakened to L^1_E-boundedness, by proving that for $1 \leq p < \infty$, E is finite dimensional if and only if every E-valued L^p_E-bounded amart converges weakly a.s.

In [14], Edgar and Sucheston used Bellow's result to show that E is finite dimensional if and only if every E-valued amart is a MIL (or a GFT). In [13] they gave counterexamples to show that Chacon and Sucheston's conditions cannot be weakened.

5.2 Relationships Between the Processes

We let E denote a separable Banach space with norm $\|\cdot\|$, and E^* denotes it's dual.

Let $(\Omega, \mathcal{B}, P, \{\mathcal{B}_n\})$ be a filtered probability space. A function $X: \Omega \to E$ is called a random variable if $\{\omega \mid X(\omega) \in B\} \in \mathcal{B}$ whenever B is open in the norm topology of E. The Banach space of integrable random variables is denoted L^1_E, with norm $\|\cdot\|_1$.

Each of the processes defined below is assumed to be adapted and in L^1_E.

The theory of integration and conditional expectation that we employ in this chapter is from Neveu ([25] p. 102 ff.).

DEFINITION 5.2.1. A process $\{X_n\}$ is called a *martingale* if $E(X_{n+1}|\mathcal{B}_n) = X_n$ for each $n \in \mathbb{N}$.

DEFINITION 5.2.2. A process $\{X_n\}$ is called a *quasimartingale* if $\sum_n E(\|E(X_{n+1}-X_n|\mathcal{B}_n)\|) < \infty$.

DEFINITION 5.2.3. A process $\{X_n\}$ is called an *amart* if the net $\{E(X_\tau), \tau \in T\}$ converges in E.

DEFINITION 5.2.4. A process $\{X_n\}$ is called a *martingale in the limit* [abbreviated MIL] if $\{Y_m\}$ converges to zero a.s. where

$$Y_m = \sup_{n \geq m} \|E(X_n - X_m|\mathcal{B}_m)\|$$

DEFINITION 5.2.5. A process $\{X_n\}$ is called an L_E^1*-martingale* if $\{y_m\}$ converges to zero where

$$y_m = \sup_{n \geq m} \|E(X_n - X_m|\mathcal{B}_m)\|_1$$

DEFINITION 5.2.6. A process $\{X_n\}$ is called a *game which gets fairer with time* [abbreviated GFT] if for each $\varepsilon > 0$ the sequence $\{y_m\}$ converges to zero where

$$y_m = \sup_{n \geq m} P[\|E(X_n - X_m|\mathcal{B}_m)\| > \varepsilon]$$

The following diagram illustrates some of the relationships between the martingale-like processes. The proofs are given, following the diagram.

DIAGRAM 5.2.7

$$\begin{tikzcd}
 & & L_E^1\text{-martingale} \arrow[dr] & \\
\text{Martingale} \arrow[r] & \text{Quasimartingale} \arrow[ur] \arrow[r] \arrow[dr] & \text{MIL} \arrow[r] & \text{GFT} \\
 & & \text{Amart} &
\end{tikzcd}$$

□

LEMMA 5.2.8. If $X \in L_E^1$ and if $\mathcal{B}_2 \supset \mathcal{B}_1$ are two sub σ-algebras of $\mathcal{B}$ then $E(E(X|\mathcal{B}_2)|\mathcal{B}_1) = E(X|\mathcal{B}_1)$.

PROOF: Let $\{X_n\}$ be a sequence of simple random variables converging in L_E^1 to X. Since $E(\cdot|\mathcal{C})$ is a continuous linear operator on L_E^1 for any sub σ-algebra $\mathcal{C}$, we get $E(X_n|\mathcal{B}_2) \to E(X|\mathcal{B}_2)$ in L_E^1, and $E(X_n - E(X_n|\mathcal{B}_2)|\mathcal{B}_1) \to E(X-E(X|\mathcal{B}_2)|\mathcal{B}_1)$ in L_E^1.

For each $n \in \mathbb{N}$ we have

$$\begin{aligned}
E(E(X_n|\mathcal{B}_2)|\mathcal{B}_1) &= E\left(E(\textstyle\sum x_k 1_{\{X_n=x_k\}}|\mathcal{B}_2)|\mathcal{B}_1\right) \\
&= E\left(\textstyle\sum x_k E(1_{\{X_n=x_k\}}|\mathcal{B}_2)|\mathcal{B}_1\right) && \text{by definition} \\
&= \textstyle\sum x_k E\left(E(1_{\{X_n=x_k\}}|\mathcal{B}_2)|\mathcal{B}_1\right) && \text{approximating } E(1_{\{X_n=x_k\}}|\mathcal{B}_2) \\
& && \text{by simple functions} \\
&= \textstyle\sum x_k E\left(1_{\{X_n=x_k\}}|\mathcal{B}_1\right) && \text{by the real case} \\
&= E(X_n|\mathcal{B}_1)
\end{aligned}$$

So $E(X_n - E(X_n|\mathcal{B}_2)|\mathcal{B}_1) = 0$ for $n \in \mathbb{N}$ and thus

$$E(X-E(X|\mathcal{B}_2)|\mathcal{B}_1) = 0 \Rightarrow E(X|\mathcal{B}_1) = E(E(X|\mathcal{B}_2)|\mathcal{B}_1)$$

□

The proof that every quasimartingale is an amart is exactly as in 3.2.11, replacing absolute value with $\|\cdot\|$. Unlike the real case, for an arbitrary Banach space Amart $\not\to$ MIL and Amart $\not\to$ L^1_E-martingale (see 5.3.2). We therefore give direct proofs that every quasimartingale is both a MIL and an L^1_E-martingale.

PROPOSITION 5.2.9. Every quasimartingale is an L^1_E-martingale.

PROOF: $$E\|E(X_n|\mathcal{B}_m)-X_m\| = E\left\|E\left(\sum_{k=m}^{n-1} E(X_{k+1}-X_k|\mathcal{B}_k)|\mathcal{B}_m\right)\right\|$$

$$\leq E\left(\sum_{k=m}^{n-1} \|E(X_{k+1}|\mathcal{B}_k)-X_k\|\right) = \sum_{k=m}^{n-1} E\|E(X_{k+1}|\mathcal{B}_k)-X_k\|$$

which converges to zero as $n \geq m \to \infty$, so $\{X_n\}$ is an L^1_E-martingale. □

PROPOSITION 5.2.10. Every quasimartingale is MIL.

PROOF: Suppose $\{X_n\}$ is a quasimartingale which is not MIL. Since it is not MIL we can find $\delta > 0$ such that

$$P[\limsup_m A_m] > \delta$$

where $A_m = \left\{\left(\sup_{n\geq m} \|E(X_n-X_m|\mathcal{B}_m)\|\right) > \delta\right\}$ $m \in \mathbb{N}$, furthermore, since $\{X_n\}$ is a quasimartingale we can choose M such that $\sum_M^\infty E\|E(X_{k+1}-X_k|\mathcal{B}_k)\| < \delta^2$.

Now choose $M < m_1 < m_2 < \dots < m_K$ so that

$$P\left[\bigcup_{j=1}^{K} A_{m_j}\right] > \delta$$

and let N be so large that

$$P\left[\bigcup_j \left(\sup_{m_j<n<N} \left\|E\left(X_n - X_{m_j} \middle| \mathcal{B}_{m_j}\right)\right\| > \delta\right)\right] > \delta$$

Now the inequality $\|E(X_n - X_m | \mathcal{B}_m)\| = \left\|E\left(\sum_{k=m}^{n-1} E(X_{k+1} - X_k | \mathcal{B}_k) \middle| \mathcal{B}_m\right)\right\|$

$$\leq E\left(\left\|\sum_{k=m}^{n-1} E(X_{k+1} - X_k | \mathcal{B}_k)\right\| \middle| \mathcal{B}_m\right) \leq E\left(\sum_{k=m}^{n-1} \|E(X_{k+1} - X_k | \mathcal{B}_k)\| \middle| \mathcal{B}_m\right)$$

tells us that for $j \in \mathbb{N}$ fixed, and $B \in \mathcal{B}_{m_j}$

$$\int_B \left(\sup_{m_j<n<N} \left\|E\left(X_n - X_{m_j} \middle| \mathcal{B}_{m_j}\right)\right\|\right) \leq \int_B \sum_{k=m_j}^{N} \|E(X_{k+1} - X_k | \mathcal{B}_k)\|$$

For $j = 1,\ldots,K$ let $B_j = A_{m_j} \cap (\bigcap_{i<j} A_{m_i}^c)$. Then $B_j \in \mathcal{B}_{m_j}$ and

$$\begin{aligned}
\delta^2 &< \sum_{j=1}^{K} \int_{B_j} \sup_{m_j<n<N} \left\|E\left(X_n - X_{m_j} \middle| \mathcal{B}_{m_j}\right)\right\| \\
&\leq \sum_{j=1}^{K} \int_{B_j} \sum_{k=m_j}^{N} \|E(X_{k+1} - X_k | \mathcal{B}_k)\| \\
&\leq E\left(\sum_{k=M}^{N} \|E(X_{k+1} - X_k | \mathcal{B}_k)\|\right) \\
&= \sum_{k=M}^{N} E\|E(X_{k+1} - X_k | \mathcal{B}_k)\| < \delta^2
\end{aligned}$$

which is a contradiction, so $\{X_n\}$ must be a MIL. $\square$

5.3 Convergence

PROPOSITION 5.3.1. The following are equivalent.

(i) Every uniformly integrable E-valued martingale converges in L_E^1.

(ii) Every uniformly integrable E-valued GFT converges in L_E^1.

PROOF: Use Subramanian's proof (3.3.7), changing absolute value to E-norm.

EXAMPLE 5.3.2 ([8]). We present an example of an E-valued amart which is uniformly integrable but does not converge in L_E^1, even though E has the Radon-Nikodym property. In light of Chatterji's result (see 5.1), we observe that we cannot replace "GFT" with "amart" in 5.3.1.

Let $E = \ell^2$, which has the Radon-Nikodym property, since it is reflexive (see 5.1). Let $\{e_n^i : n \in \mathbb{N}, 1 \leq i \leq 2^n\}$ be the standard orthonormal basis for ℓ^2 in some order. For each $n \in \mathbb{N}$ let $\{A_n^i, 1 \leq i \leq 2^n\}$ be a partition of the probability space such that $P(A_n^i) = 2^{-n}$. Let

$$X_n = \sum_{i=1}^{2^n} e_n^i \, 1_{A_n^i}$$

for each $n \in \mathbb{N}$. Observe that $\|X_n\| = 1$ everywhere so that $\{X_n\}$ is uniformly integrable. Also if $\tau \in T$,

$$\int X_\tau = \sum_{i,n} P(A_n^i \cap \{\tau = n\}) e_n^i$$

so if $\tau \geq N$, then $P[A_n^i \cap \{\tau = n\}] \leq 2^{-N}$ for all i,n. Thus

$$\|\int X_\tau\|^2 = \sum P[A_n^i \cap \{\tau = n\}]^2 \leq 2^{-N} \sum P[A_n^i \cap \{\tau = n\}] = 2^{-N}$$

so $E(X_\tau)$ converges to zero and hence $\{X_n\}$ is an amart. But $\|X_{n+1}(\omega) - X_n(\omega)\| = \sqrt{2}$ everywhere for all n, and thus $\{X_n\}$ does not converge in L_E^1.

□

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